MATH 1104
FALL 2017
LECTURE NOTES

These notes replace neither the textbook nor the lectures

Last modified: August 1, 2017

Part 1

- Systems of Linear Equations
- Elementary Row Operations
- Row Echelon Forms
- Vectors Equations
- The Matrix Equations $Ax = b$
- Solution Sets of Linear Systems
- Linear Dependence and Independence
- Linear Transformations
- The Matrix of a Linear Transformation
- Applications of Linear Systems

SYSTEMS OF LINEAR EQUATIONS

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same set of variables, say $x_1, x_2, ..., x_n$.

Example:

$$\begin{aligned} 2x_1 - x_2 - 3x_3 &= -1 \\ -2x_1 + 2x_2 + 5x_3 &= 3. \end{aligned}$$

is a linear system and $(x_1, x_2, x_3) = (1, 0, 1)$ is a solution for this system.

$(x_1, x_2, x_3) = (2, -4, 3)$ is also a solution for this system.
The set of all possible solutions is called the **solution set** of the linear system.

Two linear systems are called **equivalent** if they have the same solution set.

Example:

$$\begin{aligned} x + y &= 4 \quad | \quad & 8x + 2y &= 26 \\ x - y &= 2 \quad | \quad & 13x + 3y &= 42 \end{aligned}$$

are equivalent systems since the solution set for both of them is $\{(3, 1)\}$.

Which of the following system(s) is/are linear?

$$\begin{aligned} x^2 + y &= 7 \\ x - 5y &= 10 \end{aligned}$$

$$\begin{aligned} 3x + xy &= 6 \\ x - y &= 3 \end{aligned}$$

$$\begin{aligned} \sqrt{8}x + y &= 4 \\ 2x - 3y &= 5 \end{aligned}$$

Remark: Geometrically, solution of two linear equations in two variables is the intersection of two lines.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1: &\quad x + y = 3 \\ l_2: &\quad x - y = 1 \end{aligned}$$

Solution: $(2, 1)$ is the (unique) intersection point of l_1 and l_2. So, the system has a unique solution.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1: &\quad -x + 2y = 1 \\ l_2: &\quad x - 2y = -3 \end{aligned}$$

Solution: Lines are parallel, no intersection. So, the system has no solution.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1: &\quad 2x - y = 1 \\ l_2: &\quad -4x + 2y = -2 \end{aligned}$$

Solution: l_1 and l_2 coincide. The system has infinitely many solutions.

Each of the points

$$(x, y) = (1, 1),\ (2, 3),\ (3, 5)$$

is a solution for this system.

The general solution: $x = t$, $y = -1 + 2t$, $t \in R$.

The solution set is
$\{(x, y) \mid x = t, y = -1 + 2t,\ t \in R\}$ or $\{(t, -1 + 2t) \mid t \in R\}$ or $\{(x, 2x - 1) \mid x \in R\}$.

A system of linear equations has either

1) No solution, or
2) (Unique) Exactly one solution, or
3) Infinitely many solutions.

A system of linear equations is called **consistent** if it has at least one solution, and **inconsistent** if it has no solution.

Row Echelon Forms (REF)

A **leading entry** of a row refers to the left most non-zero entry (in a non-zero row).

Echelon form (or row echelon form):

1. Any rows consisting entirely of zeros are placed at the bottom of the matrix.

2. The first non-zero element in each row is positioned to the right of the first non-zero element in the previous row.

Example:

$$\textbf{i)}\begin{bmatrix} 2 & 1 & 6 & 4 & -2 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \textbf{ii)}\begin{bmatrix} 4 & 3 & -3 & 5 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 1 & 8 & -4 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad \textbf{iii)}\begin{bmatrix} 2 & 5 & 3 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textbf{iv)}\begin{bmatrix} 0 & 2 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 5 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 7 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & * \end{bmatrix}$$

Some terms that we use

pivot position: a position of a leading entry in an echelon form of a matrix.

pivot: a non-zero number that is in a pivot position.

pivot column: a column that contains a pivot position.

basic variable: a variable that corresponds to a pivot column.

free variable: a variable that is not a basic variable, i.e, a variable that corresponds to a non-pivot column.

Note: In an echelon form, in each column that contains a leading entry of some row, all entries below the leading entry are zero.

Reduced Row Echelon Forms (RREF)

A matrix A is said to be in reduced row echelon form if it satisfies the following conditions.
- It is in row echelon form.
- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

Examples:

$$\textbf{i)} \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \text{ (RREF)}$$

$$\textbf{ii)} \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

(RREF if $a_{13} = 0$, not RREF if $a_{13} \neq 0$).

$$\textbf{iii)} \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & -1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \text{ (not RREF)}$$

Note: $*$ can be any number.

$$\textbf{iv)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ (RREF)}$$

$$\textbf{v)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ (RREF)}$$

$$\textbf{vi)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ (not RREF, not REF)}$$

REF of a matrix is not unique:

Example:

$$\begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{ (REF)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ (REF)}$$

$$\sim \begin{bmatrix} 3 & 3 & 3 & 9 \\ 0 & 2 & 6 & 20 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{ (REF)}$$

RREF of a matrix is unique:

$$A = \begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{(RREF)}} = B$$

The matrix B is the only **RREF** of A.

Elementary Row Operations:

i) Multiply a row by a non-zero scalar: $R_2' = 5R_2$

ii) Interchange any two rows: $R_1 \longleftrightarrow R_3$

iii) Replace a row by the sum of itself and a multiple of another row: $R_3' = R_3 + 4R_2$

Example: Solve the following system.

$$\begin{aligned} x + 2y &= 4 \\ -x + 3y + 3z &= -2 \\ y + z &= 0. \end{aligned}$$

Solution: The coefficient matrix of the system is

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Its size is 3×3 (3 rows and 3 columns).
The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Its size is 3×4 (3 rows and 4 columns).
We start with the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2' = R_2 + R_1 \sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array}\right] R_3' = R_3 - 5R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] \quad (REF).$$

Since each column of the coefficient matrix has a pivot, all the variables are basic. Thus, there is a unique solution. Corresponding linear system is

$$\begin{aligned} x + 2y &= 4 \\ y + z &= 0 \\ -2z &= 2, \end{aligned}$$

which has the solution

$$\begin{aligned} &z = -1, \\ &y = -z = 1, \\ &x = -2y + 4 = -2 + 4 = 2. \end{aligned}$$

The solution is $(x, y, z) = (2, 1, -1)$.

The way that we solve the system here is called **back substitution**.

Remark: We can also find the **RREF** of the augmented matrix and use it to find the solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] R_3' = -\frac{1}{2}R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] R_2' = R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] R_1' = R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \text{(RREF)}$$

The solution is $(x, y, z) = (2, 1, -1)$.

Example: Determine whether the following linear system has a solution.

$$\begin{aligned} x + 2y + z &= 3 \\ x - y + z &= 1 \\ -2x - 4y - 2z &= 4 \end{aligned}$$

Solution: The augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -4 & -2 & 4 \end{array}\right] \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 + 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

The last row says

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 10,$$

which is imposible. So, the system does not have any solutions.

Example: Solve the following system of linear equations:

$$\begin{aligned} x+2y-3z &= 3 \\ -2x-5y+4z &= 5 \\ -5x-13y+9z &= 18 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right] \begin{array}{l} R_2^{'} = R_2 + 2R_1 \\ R_3^{'} = R_3 + 5R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array}\right] R_3^{'} = R_3 - 3R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1^{'} = R_1 + 2R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -7 & 25 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2^{'} = -R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -7 & 25 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (RREF).$$

x and y are basic variables and z is free variable.

$$x-7z=25 \Longrightarrow x=25+7z,$$
$$y+2z=-11 \Longrightarrow y=-11-2z.$$

The general solution in parametric vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25+7z \\ -11-2z \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ -11 \\ 0 \end{bmatrix} + z \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}, \quad z \in R.$$

The system is consistent, and it has infinitely many solutions.

Exercise: Solve the following system:

$$\begin{aligned} 2x + 4y - 6z &= 2 \\ y + 3z &= 5 \\ -3x - 5y + 7z &= -3 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array}\right] R_1{}' = \frac{1}{2} R_1 \sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array}\right] R_3{}' = R_3 + 3R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} R_1{}' = R_1 - 2R_2 \\ R_3{}' = R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -5 & -5 \end{array}\right] R_3{}' = -\frac{1}{5} R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1{}' = R_1 + 9R_3 \\ R_2{}' = R_2 - 3R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

The solution of the system: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$

Warning: A zero row in an augmented matrix always does not mean infinitely many solutions:

- $\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$: There are no solutions.

- $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$: There are no solutions.

- $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$: There is a unique solution.

- $\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 5 & 3 \\ 0 & 1 & 3 & 2 & -2 & 4 \\ 0 & 0 & 0 & 4 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$: There are infinitely many solutions.

Remark: We can write the system

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \quad (*) \text{ (linear system)} \\ -5x - 13y + 9z &= 18 \end{aligned}$$

in the form of

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -5 & 4 \\ -5 & -13 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \quad (**) \text{ (matrix equation)}$$

or

$$x \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + y \begin{bmatrix} 2 \\ -5 \\ -13 \end{bmatrix} + z \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} (***) \text{ (vector equation).}$$

Solving the linear system (*) is the same as solving the matrix equation (**), or vector equation (***).

Example: Find the value of the constant k such that the following system has
i) no solution,
ii) infinitely many solutions,
iii) unique solution.

$$\begin{aligned} x+2y-z &= 1 \\ -2x-3y+2z &= -1 \\ -5x-8y+5z &= k \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ -2 & -3 & 2 & -1 \\ -5 & -8 & 5 & k \end{array}\right] \begin{array}{l} R_2' = R_2+2R_1 \\ R_3' = R_3+5R_1 \end{array} \sim \left[\begin{array}{rrr|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5+k \end{array}\right] \begin{array}{l} R_1' = R_1-2R_2 \\ R_3' = R_3-2R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3+k \end{array}\right].$$

i) $k \neq -3$. **ii)** $k=-3$.
iii) There is no k such that the system has a unique solution.

Example: Solve the **homogeneous** system:

$$\begin{aligned} x+3y-2z-w &= 0 \\ -2x-5y+4w &= 0 \\ x+4y-6z+w &= 0 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ -2 & -5 & 0 & 4 & 0 \\ 1 & 4 & -6 & 1 & 0 \end{array}\right] \begin{array}{l} R_2' = R_2+2R_1 \\ R_3' = R_3-R_1 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -4 & 2 & 0 \end{array}\right] R_3' = R_3-R_2$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\begin{aligned} y &= 4z-2w, \\ x &= -3y+2z+w = -3(4z-2w)+2z+w = -10z+7w. \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z+7w \\ 4z-2w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix}; \quad z,w \in R.$$

Example: Solve the **non-homogeneous** system:

$$\begin{aligned} x + 3y - 2z - w &= 2 \\ -2x - 5y + 4w &= 3 \\ x + 4y - 6z + w &= 9 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ -2 & -5 & 0 & 4 & 3 \\ 1 & 4 & -6 & 1 & 9 \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 - R_1 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 1 & -4 & 2 & 7 \end{array}\right] R_3' = R_3 - R_2$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$y = 4z - 2w + 7,$$

$$x = -3y + 2z + w + 2 = -3(4z - 2w + 7) + 2z + w + 2 = -10z + 7w - 19.$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z + 7w - 19 \\ 4z - 2w + 7 \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}; \quad z, w \in R.$$

$$\left\{ \begin{array}{c} \text{general solution} \\ \text{of the corresponding} \\ \text{homogeneous system} \end{array} \right\} = v_h = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

$$\left\{ \begin{array}{c} \text{a particular solution of the} \\ \text{non-homogeneous system} \end{array} \right\} = p = \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

The general solution of a non-homogeneous system is $\boxed{X = v_h + p}$.

Example: Describe all solutions of the homogeneous equation $Ax = 0$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 6 & 0 & 8 & -1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \quad \begin{array}{l} (x_1,\ x_3,\ x_6 \text{ are basic variables}) \\ (x_2,\ x_4,\ x_5 \text{ are free variables}) \end{array}$$

The general solution (in parametric vector form) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -6x_2 - 8x_4 + x_5 \\ x_2 \\ 3x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad x_2, x_4, x_5 \in R.$$

Linear Combinations of Vectors

Definition: If $v_1, v_2, v_3, ..., v_p$ are vectors in R^n, and $c_1, c_2, c_3, ..., c_p$ are scalars, then the vector x defined by

$$x = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_pv_p$$

is called a **linear combination** of $v_1, v_2, v_3, ..., v_p$.

Example: Determine if the vector $b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the vectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \; v_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}.$$

Solution:

$$b \in \text{ Span}\{v_1, v_2, v_3\} \Longleftrightarrow xv_1 + yv_2 + zv_3 = b$$

has a solution.

$$\left[\begin{array}{rrr|r} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent if and only if $b \in \text{Span}\{v_1, v_2, v_3\}$.
The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-5z \\ 3-4z \\ z \end{bmatrix}, z \in R,$$

which means that for any scalar z,

$$(2-5z)v_1 + (3-4z)v_2 + zv_3 = b.$$

For $z = -1$: $7v_1 + 7v_2 - v_3 = b$.

For $z = 0$: $-2v_1 + 3v_2 + 0v_3 = b \Longrightarrow -2v_1 + 3v_2 = b$.

For $z = 1$: $-3v_1 - v_2 + v_3 = b$.

For $z = 3$: $-8v_1 - 5v_2 + 2v_3 = b$.

The set of all linear combinations of $v_1, v_2, ..., v_p$ is called the **span** of $v_1, v_2, v_3, ..., v_p$ and denoted by $\text{Span}\{v_1, v_2, v_3, ..., v_p\}$.

- Scalar multiples of v_i are in $\text{Span}\{v_1, v_2, ..., v_p\}$.
- Zero vector is in $\text{Span}\{v_1, v_2, ..., v_p\}$.
- $\text{Span}\{v_1\} = \{cv_1 | c \in R\}$.
- $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is the x-axis.

Example: Let

$$b = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}.$$

For what value(s) of h, b is in $\text{Span}\{v_1,\ v_2\}$?

Solution:

$$\left[\begin{array}{cc|c} v_1 & v_2 & b \end{array}\right] = \left[\begin{array}{rr|r} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{array}\right] R_3{}' = R_3 + 2R_1$$

$$\sim \left[\begin{array}{rr|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h-5 \end{array}\right] R_3{}' = R_3 - 3R_2$$

$$\sim \left[\begin{array}{rr|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{array}\right].$$

The system is consistent if and only if $2h + 4 = 0$. So, b is in $\text{Span}\{v_1, v_2\}$ if and only if $h = -2$.

Note: If $h = -2$, then we have

$$\begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix} = -8 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}.$$

Example: Solve the matrix equation

$$AX = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -2 \\ -2 & 3 & 3 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & 7 & -2 & 1 \\ -2 & 3 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -16 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -14 \end{array}\right].$$

The solution of the system: $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16 \\ 3 \\ -14 \end{bmatrix}$.

Theorem (or Fact): The following statements are equivalent for any $m \times n$ matrix A.

- The equation $Ax = b$ has a solution for each b in R^m,.
- Each $b \in R^m$ is a linear combination of the columns of A.
- A has a pivot position in every row.
- The columns of A span R^m.

Linear Dependence and Independence

Definition: A set of vectors $\{v_1, v_2, ..., v_k\}$ in R^n is said to be linearly independent if the vector equation

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

has only the trivial solution, i.e, $c_1 = c_2 = \cdots = c_k = 0$.

The set $\{v_1, v_2, ..., v_k\}$ in R^n is said to be linearly dependent if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$.

Example: Decide if the set of vectors $\left\{ \begin{bmatrix} 12 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

Solution: We consider the vector equation

$$c_1 \begin{bmatrix} 12 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Take $c_1 = 1$ and $c_2 = 3$. Then

$$\begin{bmatrix} 12 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, v_1 and v_2 are linearly dependent.

A set of two vectors $\{v_1,\ v_2\}$ is linearly dependent if and only if one is a scalar multiple of the other.

Example: Decide if the following vectors are linearly independent in R^2.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Solution: Consider the vector equation $xv_1 + yv_2 + zv_3 = 0$.

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix}, \ z \in R.$$

Choose $z = 1$. Then

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The vectors are linearly dependent.

Note that none of the above vectors is a multiple of one of the other vectors.

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ v_2, ..., v_p\}$ in R^n is linearly dependent if $p > n$.

- In R^2, the maximum number of linearly independent vectors is 2.
- In R^3, the maximum number of linearly independent vectors is 3.

Exercise: Are the following vectors linearly independent in R^3?

$$\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

Example: Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}$$

Solution: Consider the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$.

$$\left[\begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & 6 & -6 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right].$$

Since each column is a pivot column, there is a unique solution which is

$$x_1 = 0, \ x_2 = 0, x_3 = 0.$$

So, the given vectors are linearly independent.

Example: Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 7 \\ -2 \\ -6 \end{bmatrix} \in R^4.$$

Solution: Consider the equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$.

$$\left[\begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 4 & 3 & 0 \\ -1 & -7 & 7 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then, $c_1 = c_2 = c_3 = 0$. The vectors are linearly independent.

Remarks: We put the given vectors in a matrix A as columns.
Then we find an REF of A, say B.
• If B has a pivot position in each column, then the vectors are linearly independent.
• If B has a non-pivot column, then the vectors are linearly dependent.

Examples:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 5 & 7 \\ 0 & 0 & 3 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

• Columns of A are linearly independent.
• Columns of A do not span R^3.

• Columns of B are linearly dependent.
• Columns of B span R^3.

• Columns of C are linearly independent.
• Columns of C span R^3.

• Columns of D are linearly dependent.
• Columns of D do not span R^4.

Example: Decide if the following vectors are linearly dependent.
If yes, express one of the vectors as a linear combination of the other vectors.

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 0 & 4 \\ -1 & 0 & 3 & -1 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a non-pivot column, the vectors are linearly dependent. Note that x_1, x_2 and x_3 are basic variables, and x_4 is a free variable. The general solution of the matrix equation $Ax = b$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_4 \\ 0 \\ -x_4 \\ x_4 \end{bmatrix}, \ x_4 \in R.$$

If we choose $x_4 = 1$, then we have: $-4v_1 + 0v_2 - v_3 + v_4 = 0$. So, we have

$$v_4 = 4v_1 + v_3, \ v_1 = -\frac{1}{4}v_3 + \frac{1}{4}v_4, \ v_3 = -4v_1 + v_4.$$

The vector v_2 cannot be expressed as a linear combination of the vectors v_1, v_3 and v_4.

Remark: A set S with two or more vectors is linearly dependent $\Longleftrightarrow$ at least one of the vectors in S is expressible as a linear combination of other vectors in S.

Example: Find the value(s) of h for which the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

are linearly dependent.
Solution:

$$\begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 1 \\ -3 & 1 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 7+h \end{bmatrix}.$$

For $h = -7$, the vectors are linearly dependent.

If $h \in R$ and $h \neq -7$, then the vectors are linearly independent.

Remarks:

- A set S with two or more vectors is linearly independent $\Longleftrightarrow$ no vector in S is expressible as a linear combination of other vectors in S.
- Zero vector is linearly dependent.
  Any set of vectors containing zero vector is linearly dependent.
- Any set $\{v_1, v_2, ..., v_k\}$ in R^n is linearly dependent if $k > n$, i.e,
  # of vectors > # of entries in one vector.
- Columns of a matrix A are linearly independent $\Longleftrightarrow$ $AX = 0$ has only the trivial solution.

Exercise: Find the value(s) of h for which the vectors

$$\begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ h \\ -8 \end{bmatrix}$$

are linearly dependent.

Solution: $\begin{bmatrix} 1 & -3 & 4 \\ -5 & 8 & h \\ -2 & 6 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & -7 & 20+h \\ 0 & 0 & 0 \end{bmatrix}.$

Vectors are linearly dependent for all h in R.

Linear Transformations

Definition: A transformation (or mapping) T from R^n to R^m is called a linear transformation if satisfies the following two conditions:

i) $T(u+v) = T(u) + T(v)$ for all the vectors $u, v \in R^n$.

ii) $T(cu) = cT(u)$ for all the vectors $u \in R^n$ and all the scalars $c \in R$.

Remark: R^n is called the domain of T, and R^m is called the codomain of T.

Example: Let $T{:}R^2 \to R^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ x+y \end{bmatrix}.$$

Let us show that T is a linear transformation:
(i)

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} x+w \\ y+z \end{bmatrix}\right) = \begin{bmatrix} y+z \\ x+w \\ x+w+y+z \end{bmatrix}.$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} w \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ x+y \end{bmatrix} + \begin{bmatrix} z \\ w \\ z+w \end{bmatrix} = \begin{bmatrix} y+z \\ x+w \\ x+y+z+w \end{bmatrix}.$$

ii)

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ cx \\ cx+cy \end{bmatrix}$$

$$cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c\begin{bmatrix} y \\ x \\ x+y \end{bmatrix} = \begin{bmatrix} cy \\ cx \\ cx+cy \end{bmatrix}.$$

By i) and ii) T is a linear transformation.

Remark 1: Two conditions for a linear transformation can be written as one condition:

$$T(cu + dv) = c\,T(u) + d\,T(v)$$

for all $u,\ v \in R^n$, and for all scalars c and d.

Remark 2: For any linear transformation T,

$$T(0) = T(0u) = 0T(u) = 0.$$

Example: Let $T{:}R^2 \to R^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+1 \end{bmatrix}.$$

Show that T is not a linear transformation.
Solution:

$$\begin{aligned} T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) &= T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ cx+1 \end{bmatrix}, \\ cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= c\begin{bmatrix} y \\ x+1 \end{bmatrix} = \begin{bmatrix} cy \\ c(x+1) \end{bmatrix} = \begin{bmatrix} cy \\ cx+c \end{bmatrix}. \end{aligned}$$

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \text{ if } c \neq 1.$$

Example: For any $m \times n$ matrix A, the matrix transformation $T : R^n \longrightarrow R^m$ given by

$$T(X) = AX$$

is a linear transformation.

Solution: If A is an $m \times n$ matrix, u and v are vectors in R^n, and c is a scalar, then:

$$\begin{aligned} A(u+v) &= Au + Av \\ A(cu) &= c(Au). \end{aligned}$$

For simplicity, take $m = 3$ and $n = 2$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$\begin{aligned}
A(u+v) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11}(u_1+v_1)+a_{12}(u_2+v_2) \\ a_{21}(u_1+v_1)+a_{22}(u_2+v_2) \\ a_{31}(u_1+v_1)+a_{32}(u_2+v_2) \end{bmatrix} \\
&= \begin{bmatrix} a_{11}u_1+a_{12}u_2 \\ a_{21}u_1+a_{22}u_2 \\ a_{31}u_1+a_{32}u_2 \end{bmatrix} + \begin{bmatrix} a_{11}v_1+a_{12}v_2 \\ a_{21}v_1+a_{22}v_2 \\ a_{31}v_1+a_{32}v_2 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \\
&= Au + Av.
\end{aligned}$$

$$\begin{aligned}
A(cu) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_{11}cu_1+a_{12}cu_2 \\ a_{21}cu_1+a_{22}cu_2 \\ a_{31}cu_1+a_{32}cu_2 \end{bmatrix} \\
&= c \begin{bmatrix} a_{11}u_1+a_{12}u_2 \\ a_{21}u_1+a_{22}u_2 \\ a_{31}u_1+a_{32}u_2 \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = c(Au)
\end{aligned}$$

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ -9 & 8 & 7 \end{bmatrix}, \ u = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Compute Au, Av, $A(u+v)$, and $A(5u)$.
Solution:

$$Au = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ -9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+8-9 \\ 4-20-18 \\ -9+32-21 \end{bmatrix} = \begin{bmatrix} 0 \\ -34 \\ 2 \end{bmatrix}.$$

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ -9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2+6 \\ 0-5+12 \\ 0+8+14 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 22 \end{bmatrix}.$$

$$\begin{aligned} A(u+v) &= A(u)+A(v) \\ &= \begin{bmatrix} 0 \\ -34 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \\ 22 \end{bmatrix} = \begin{bmatrix} 8 \\ -27 \\ 24 \end{bmatrix}. \end{aligned}$$

$$A(5u) = 5(Au) = 5 \begin{bmatrix} 0 \\ -34 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -170 \\ 10 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let $T : R^2 \longrightarrow R^3$ defined by $T(X) = AX$.

a) Find $T(u)$.

b) Find a vector X in R^2 such that $T(X) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

c) Is there another vector $Y \neq X$ such that $T(Y) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$.

d) Is $W = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ in the range of T?

Solution: a)

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}.$$

b) We need to find a vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

Corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right].$$

Then, $x_2 = -2$ and $x_1 = -1 - 2x_2 = 3$. Thus, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

c) Since the system

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$

has a unique solution, the vector $X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is the only vector such that

$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

d) We need to check whether the equation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

has a solution:

$$\left[\begin{array}{rr|r} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right],$$

which is inconsistent.

So, $W = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is not in the range of T.

The matrix of a Linear Transformation

We have seen that for any $m \times n$ matrix A, the transformation $T : R^n \longrightarrow R^m$ defined by $T(X) = AX$ is a linear transformation.

Conversely, for any linear transformation $T : R^n \longrightarrow R^m$ there is a unique $m \times n$ matrix A such that $T(X) = AX$.
The matrix A is called the **standard matrix** for T.

Examples:

$$T : R^2 \longrightarrow R^2, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{A_{2\times 2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T : R^2 \longrightarrow R^3, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ -x \\ 2x+y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}}_{A_{3\times 2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T : R^3 \longrightarrow R^2, \quad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y-3z \\ 4x-y+5z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 5 \end{bmatrix}}_{A_{2\times 3}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition: Standard basis for R^n is given by

$$\mathcal{B} = \{e_1, e_2, ..., e_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}.$$

Let $T : R^n \longrightarrow R^m$ be a linear transformation such that

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, ..., T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix A given by

$$A_T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

is the **standard matrix** for T.

Example: Find the standard matrix of the linear transformation $T : R^3 \longrightarrow R^4$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ 3x - 4y + 5z \\ y + z \\ -3x + 5y - 4z \end{bmatrix}.$$

Solution:

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \quad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix},$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \\ 1 \\ -4 \end{bmatrix}.$$

So, the standard matrix of T is

$$A_T = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}. \text{ Then, } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Example: Reflection in the $x-$ axis is a linear transformation:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Example: Let $T : R^2 \to R^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

T rotates the vectors in R^2 $90°$ counterclockwise about the origin.

Examples (Rotations): A rotation in the plane is a linear transformation: Let $T : R^2 \to R^2$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ be any point in R^2. Then,

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} + y \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The linear transformation $R_\varphi : R^2 \to R^2$

$$R_\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

rotates the vectors counterclockwise by angle φ.

For $\varphi = \pi/2$: $R_{\frac{\pi}{2}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$

For $\varphi = \pi/4$: $R_{\frac{\pi}{4}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$

For $\varphi = \pi/3$: $R_{\frac{\pi}{3}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x - \sqrt{3}y \\ \sqrt{3}x + y \end{bmatrix}.$

For $\varphi = \pi$: $R_\pi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$

For $\varphi = 2\pi$: $R_{2\pi}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$

Horizontal shear and Vertical shear: The linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix}$$

is called a **horizontal shear** (or $x-$shear).
Similarly, the linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ax+y \end{bmatrix}$$

is called a **vertical shear** (or $y-$shear) for any number a.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

The transformation $T : R^2 \longrightarrow R^2$ given by $T(X) = AX$ is a positive $x-$shear.

Find $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$, and $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

The linear transformation $T : R^2 \to R^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} x \\ y \end{bmatrix}$$

is called a **contraction** if $0 \le r \le 1$ and a **dilation** if $r > 1$.

Definition: Let $T : R^n \longrightarrow R^m$ be a linear transformation.

(1) If for each vector b in R^m there is at least one vector X in R^n such that $T(X) = b$, then T is said to be **onto** R^m.

(2) If each vector b in R^m is the image of at most one vector X in R^n, then T is said to be **one-to-one**.

Example: Let $T : R^3 \longrightarrow R^2$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix},\quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\4\end{bmatrix},\quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\6\end{bmatrix}.$$

a) Is T onto? Explain your answer.

b) Is T one-to-one? Explain your answer.

Solution:

$$A=\begin{bmatrix} T(e_1) & T(e_2) & T(e_3)\end{bmatrix}=\begin{bmatrix}1&2&3\\2&4&6\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&0&0\end{bmatrix}.$$

a) Since A does not have a pivot position in each row, T is not onto.

b) $AX = 0$ has infinitely many solutions. So, T is not one-to-one.

Theorem: Let $T : R^n \longrightarrow R^m$ be a linear transformation, and let $A_{m\times n}$ be the standard matrix of T.

(1) T is one-to-one $\Longleftrightarrow$ $T(X) = 0$ has only the trivial solution.

$\Longleftrightarrow$ Columns of A are linearly independent.

$\Longleftrightarrow$ Each column of A has a pivot position.

(2) T is onto $\Longleftrightarrow$ Columns of A span R^m.

$\Longleftrightarrow$ Each row of A has a pivot position.

$\Longleftrightarrow$ For each $b \in R^m$, $AX = b$ is consistent.

Example: $T : R^3 \longrightarrow R^4$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ x_1 - x_2 \\ x_2 + x_3 \\ x_1 - x_2 \end{bmatrix}.$$

i) Is T one-to-one? Explain your answer.

ii) Is T onto? Explain your answer.

Solution:

$$A_T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

i) Since each column of REF of A_T has a pivot position, T is one-to-one.

ii) Since each row of REF of A_T does not have a pivot position, T is not onto.

Remark: If $T : R^n \longrightarrow R^m$ and $m > n$, then T cannot be onto.

Example: Let $T : R^4 \longrightarrow R^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - 3x_4 \\ -2x_1 - 4x_2 + x_3 + 5x_4 \\ x_1 + 5x_2 + x_3 - 4x_4 \end{bmatrix}.$$

i) Is T one-to-one? Explain your answer.

ii) Is T onto? Explain your answer.

Solution:

$$A_T = \begin{bmatrix} 1 & 2 & 0 & -3 \\ -2 & -4 & 1 & 5 \\ 1 & 5 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

i) Since REF of A_T has a non-pivot column, T is not one-to-one.

ii) Since each row of REF of A_T has a pivot position, T is onto.

Remark: If $T : R^n \longrightarrow R^m$ and $n > m$, then T cannot be one-to-one.

Remark: If $T : R^n \longrightarrow R^m$ is a linear transformation and $\{v_1, v_2, v_3\}$ is a linearly dependent set in R^n, then $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent in R^m.

Example: The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix},\ v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

are linearly dependent since $v_3 = -2v_1 + v_2$.
For any linear transformation $T : R^3 \longrightarrow R^m$

$$T(v_3) = T(-2v_1 + v_2) = -2T(v_1) + T(v_2).$$

Thus, the vectors

$$T(v_1),\ T(v_2),\ T(v_3)$$

are also linearly dependent.

Remark: If $\{v_1, v_2, v_3\}$ is a linearly independent set, then $\{T(v_1), T(v_2), T(v_3)\}$ does not need to be linearly independent.

Example: Consider the linear transformation $T : R^3 \longrightarrow R^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 4x_2 + 6x_3 \end{bmatrix}.$$

The vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent but

$$T(e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad T(e_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, T(e_3) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

are linearly dependent.