Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2322 B
Midterm Exam 1 Version 1

Professor: Hai Yan Liu(Jack) **October 3, 2019**

Duration: 80 minutes

Student Number:________________________

Name: ______________________________

1. Explain clearly your work: if there is a method or a theorem that you are using to solve the problem, quote it on your solution!
2. Books, notes, etc. are not allowed.
3. Only the faculty approved calculators are allowed.
4. If you write on the back of a page, please put a note.
5. Please turn off or put on silent your cell-phones.
6. DO NOT FORGET TO WRITE YOUR FIRST NAME, LAST NAME AND UNIVERSITY ID: PAPERS MISSING SUCH INFORMATION WILL NOT BE MARKED
7. Good luck/Bonne chance

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada

585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc/Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

I. [5 points] Find and classify the critical points of the function

$$f(x, y) = x^3 - x^2y + y^2 - x^2.$$

Solution: $f_x = 3x^2 - 2xy - 2x$

$f_y = -x^2 + 2y$

First. Let $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \iff \begin{cases} 3x^2 - 2xy - 2x = 0 & ① \\ -x^2 + 2y = 0 & ② \end{cases}$ ②. $2y = x^2$

③ put in ① we have $3x^2 - x^3 - 2x = 0$ $-x(x^2 - 3x + 2) = 0$

$x = 0$, $x = 1$, $x = 2$

$y = 0$ $y = \frac{1}{2}$. $y = 2$.

you have $(0, 0)$, $(1, \frac{1}{2})$, $(2, 2)$. 3 critical points

next, think about second derivative

$f_{xx} = 6x - 2y - 2$

$f_{yy} = 2$.

$f_{xy} = -2x$

at $(0, 0)$. $D = f_{xx} f_{yy} - f_{xy}^2 = -2 \times 2 - 0^2 = -4 < 0$

$(0, 0)$ is a saddle point.

at $(1, \frac{1}{2})$. $f_{xx} = 6 \times 1 - 2 \times 1/2 - 2 = 3$ $f_{xy} = -2 \times 1 = -2$

$D = 3 \times 2 - (-2)^2 = 6 - 4 = 2 > 0$, $f_{xx} = 3 > 0$.

$(1, \frac{1}{2})$ is a local minmum.

at $(2, 2)$. $f_{xx} = 6 \times 2 - 2 \times 2 - 2 = 6$, $f_{xy} = -2 \times 2 = -4$

$D = 6 \times 2 - (-4)^2 = 12 - 16 = -4 < 0$

$(2, 2)$ is a saddle point.

II. [5 points] Find the global maximum and minimum of the function

$$f(x, y) = x^2 + y^2 + x^2 y - 3$$

on the closed square with vertices $(2, 2), (-2, 2), (-2, -2), (2, -2)$.

First, find the critical point.

$f_x = 2x + 2xy$ when $\begin{cases} 2x + 2xy = 0 \\ 2y + x^2 = 0 \end{cases}$

$f_y = 2y + x^2$

$2x(1+y) = 0$, $x = 0$, $y = -1$, therefore critical points

$y = 0$, $x = \pm\sqrt{2}$.

$(0, 0)$, $(-\sqrt{2}, -1)$, $(\sqrt{2}, -1)$

$f_{xx} = 2 + 2y$, $f_{xy} = 2x$, $f_{yy} = 2$.

at $(0, 0)$. $f_{xx} = 2$, $f_{yy} = 2$. $f_{xy} = 0$.

$D = f_{xx} f_{yy} - f_{xy}^2 = 2 \times 2 - 0 > 0$ $f_{xx} = 2 > 0$

$(0, 0)$ is local $\boxed{\text{minimum.}}$ $f(0, 0) = \boxed{-3}$

at $(-\sqrt{2}, -1)$ $f_{xx} = 2 + 2 \times (-1) = 0$, $f_{xy} = -2\sqrt{2}$

$D = f_{xx} f_{yy} - f_{xy}^2 = 0 \times 2 - (-2\sqrt{2})^2 < 0$

$(-\sqrt{2}, -1)$ is a saddle point. $f(-\sqrt{2}, -1) = 2 + 1 - 2 - 3 = \boxed{-2}$

at $(\sqrt{2}, -1)$. $f_{xx} = 2 + 2(-1) = 0$, $f_{xy} = 2\sqrt{2}$.

$D = f_{xx} f_{yy} - f_{xy}^2 = 0 \times 2 - (2\sqrt{2})^2 < 0$

is a saddle point.

discuss on the boundary:

at $x = -2$. $f(-2, y) = 4 + y^2 + 4y - 3 = y^2 + 4y - 1$

$f'(y) = 2y + 4 = 0$, $y = -2$.

We need to discuss

$y=-2$. $f(-2, y) = 4+y^2+4y-3 = y^2+4y+1 = (-2)^2-8+1 = \boxed{-3}$

$y=2$, $f(-2, 2) = 2^2+8+1 = \boxed{13}$

at $x=2$. $f(2, y) = y^2+4y+1$.

at $y=-2$. $f(+2, -2) = -3$.

$y=2$ $f(2, 2) = \boxed{13}$

next we discuss when $y=-2$

$$f(x, -2) = x^2+(-2)^2+x^2(-2)-3 = -x^2+1$$

when $x=0$, $f(0, -2) = 1$. local minimum.

when $x=-2$, $f(-2, -2) = \boxed{-3.}$

Finally we discuss when $y=2$.

$$f(x, 2) = x^2+(-2)^2+x^2(2)-3 = 3x^2+1$$

when $x=0$, $f(0, 2) = 1$

$x=-2$ $f(-2, 2) = 3\times(-2)^2+1 = \boxed{13}$

$x=2$. $f(2, 2) = 3\times(2)^2+1 = \boxed{13}$

Therefore global minimum is $\boxed{-3}$ and global maximum $\boxed{13}$

III. [5 points] Use the method of Lagrange multipliers to find the absolute minimum and maximum of the function

$$f(x,y) = x + y^2$$

defined on the constraint

$$x^2 + y^2 = 1$$

$x^2+y^2-1=0$. $g(x,y)=x^2+y$

construct: $F(x,y) = f(x,y) + \lambda g(x,y)$.

$$= x + y^2 + \lambda(x^2+y^2-1).$$

$$\begin{cases} F_x = 1 + 2\lambda x = 0 \\ F_y = 2y + 2\lambda y = 0 \\ F_\lambda = x^2+y^2-1 = 0 \end{cases} \iff \begin{cases} 1+2\lambda x = 0 \\ 2y(1+\lambda) = 0 \\ x^2+y^2-1 = 0 \end{cases}$$

when $y=0$, $x=\pm 1$, $(-1, 0)$. $(1, 0)$.

when $\lambda = -1$ $x = 1/2$. $y = \pm\frac{\sqrt{3}}{2}$. $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

at $(-1, 0)$ $f(-1, 0) = -1$ $f(1, 0) = 1$

at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. $f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{1}{2} + (\frac{\sqrt{3}}{2})^2 = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$

at $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. $f(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{1}{2} + (-\frac{\sqrt{3}}{2})^2 = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$.

the absolute minimum is -1, and ~~no~~ absolute maximum is $\boxed{\frac{5}{4}}$

IV. [5 points] Find the critical points of the function

$$f(x,y) = xye^{-(x^2+y^2)}$$

(you do not have to classify them)

$$f_x = ye^{-(x^2+y^2)} + xye^{-(x^2+y^2)}(-2x)$$
$$= e^{-(x^2+y^2)}(y - 2x^2y)$$
$$f_y = xe^{-(x^2+y^2)} + xye^{-(x^2+y^2)}(-2y)$$
$$= e^{-(x^2+y^2)}(x - 2xy^2)$$

when $\begin{cases} f_x = 0 \\ f_y = 0, \end{cases}$ simplify them we get

$$\begin{cases} y - 2x^2y = 0 \\ x - 2xy^2 = 0 \end{cases} \iff \begin{cases} y(1-2x^2) = 0. \\ x(1-2y^2) = 0. \end{cases}$$

when $y = 0$. $x = 0$. when $1 - 2x^2 = 0$, $x = \pm\frac{1}{\sqrt{2}}$

$y = \pm\frac{1}{\sqrt{2}}$. So ~~they~~ their critical points are

$(0, 0)$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$., $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

V. [5 points] True or false? Please provide a quick explanation to motivate your conclusions.

1. The function $f(x, y) = |x| + |y|$ admits partial derivatives at $(x, y) = (0, 0)$.

False, $f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h}$

$= \lim_{h \to 0} \frac{|h| + |0| - |0| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$.

does not exists.

Same as $f_y(0, 0)$.

2. We can use the method of Lagrange multipliers, as learned in class, to find the extreme values of the function $f(x, y) = e^{x^2 + \sin(xy)}$ along the constraint

$$C = \{(x, y) \in R^2 \mid y = |x|\}.$$

False. since. we need. $F_x(x.y) = f(x,y) + \lambda g(x.y)$. partical derivative exist, but $g(x.y) = y - |x|$. does not have partical derivative.

3. Let $f(x, y) = \arctan(\ln(x^3 y e^{x^2 y \sin(x^2 + y^8)})) + \frac{x}{x + \ln(1 + xy)}$. On its domain of definition, $f_{xy} = f_{yx}$.

Yes, we know on its domain $f(x.y)$ derivative is continuous, since it is continuous, then we have

$$f_{xy} = f_{yx}$$