# MATH3705 Notes - By Eric Hua

## Contents

# 1 LAPLACE TRANSFORM

## 1.1 Introduction

### 1.1.1 Definition

The Laplace Transform is widely used in engineering applications, such as solving linear ordinary differential equations. It transforms the equation in "t-space" to one in "s-space". This makes the problem much easier to solve.

Let $f(t)$ be a function defined on $[0, \infty)$. We may assume $f(t) = 0$ when $t < 0$. The Laplace transform (LT) of $f(t)$ is the function $F(s)$, defined by:

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt, \quad s > 0.$$

This is named for Pierre-Simon Laplace, one of the best French mathematicians in the mid-to-late 18th century.The LT transforms functions of $t$ to functions of another variable $s$.

If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is the **inverse Laplace transform** of $F(s)$:

$$f(t) = L^{-1}\{F(s)\}.$$

**Proposition 1.** *The LT and $L^{-1}$ are **linear**:*

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

$$L^{-1}\{aF(s) + bG(s)\} = aL^{-1}\{F(s)\} + bL^{-1}\{G(s)\}$$

For example,

$$L\{1\} = \int_0^{\infty} e^{-st}\, dt = \left\{-\frac{1}{s}e^{-st}\right\}_0^{\infty} = \frac{1}{s}.$$

Since $\int e^{bt} \sin at dt = \frac{e^{bt}}{a^2+b^2}(b \sin at - a \cos at)$, we have

$$L\{\sin at\} = \int_0^{\infty} e^{-st} \sin at\, dt = \left\{\frac{e^{-st}}{a^2+s^2}(-s \sin at - a \cos at)\right\}_0^{\infty} = \frac{a}{s^2+a^2}.$$

$$L\{e^{at}\} = \int_0^{\infty} e^{(a-s)t}\, dt = \left\{-\frac{1}{s-a}e^{(a-s)t}\right\}_0^{\infty} = \frac{1}{s-a}.$$

| Examples of Laplace Transforms | |
|---|---|
| $f(t)$ for $t \geq 0$ | $L\{f(t)\}$ |
| $1$ | $\frac{1}{s}$ |
| $e^{at}, \quad s > a$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$ |
| $\sin at$ | $\frac{a}{s^2+a^2}$ |
| $\cos at$ | $\frac{s}{s^2+a^2}$ |
| $f(at)$ | $\frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$ |

### 1.1.2 Existence

For which functions $f$ is the LT actually defined on? We want the indefinite integral to converge, of course.

A function $f(t)$ is piecewise continuous on a finite interval $[a, b]$ if $[a, b]$ can be subdivided into a finite number of subintervals such that f is continuous in each of the subintervals, and it approaches a finite limit when t approaches an end of any of the subintervals. Function f is piecewise continuous on an infinite interval if it is piecewise continuous on any finite subinterval of its domain.

A function $f(t)$ is of **exponential order** $\alpha$ if there exist constants $t_0$ and $M$ such that

$$|f(t)| < Me^{\alpha t}, \quad \text{for all } t > t_0.$$

For example, $t^n$ and $e^t$ are of exponential order, $e^{t^2}$ is not of exponential order.

**Theorem 1.** *Suppose that $f(t)$ is piecewise continuous and of exponential order with*

$$|f(t)| < Me^{\alpha t}, \quad \textit{for all } t > t_0.$$

*Then the Laplace transform of $f(t)$ exists for all $s > \alpha$.*

Proof.

$$|F(s)| \leq M \int_0^\infty e^{(\alpha - s)t}\, dt = \frac{M}{s - \alpha} < \infty.$$

Remark. The condition in this theorem is sufficient but not necessary. For example, the function $f(t) = t^n, -1 \leq n < 0$ is not piecewise continuous. But the Laplace transform of $f(t)$ exists.

Remark. If $f(t)$ satisfies the conditions in the theorem above, then $F(s) \to 0$ when $s \to \infty$.

**Example 1.** $L\left\{e^{t^2}\right\}$ *does not exist.*

Proof.

$$L\left\{e^{t^2}\right\} \geq e^{-s^2/4} \int_0^\infty e^{(t-s/2)^2}\,dt \geq e^{-s^2/4} \int_0^\infty 1\,dt = \infty.$$

### 1.1.3 The Gamma Function

To find Laplace transform of $t^n$ where n is not an integer, we need the Gamma function:

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \quad (p > 0).$$

**Properties:**

- $\Gamma(1) = 1$.
- It is divergent when $p = 0$.
- $\Gamma(p+1) = p\Gamma(p)$ for any $p$ by integration by parts, except for 0, -1, -2, -3, .
- $\Gamma(n+1) = n!$

**Example 2.** $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

**Example 3.** $\Gamma(\frac{2k+1}{2}) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}$.

$\Gamma(-\frac{2k-1}{2}) = \frac{2^k}{(2k-1)!!}\sqrt{\pi}$.

**Theorem 2.** *For any* $p > -1$, $L\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}$.

Proof. Let $x = st$.

**Example 4.** $L\{t^{0.5}\} = \frac{\Gamma(0.5+1)}{s^{0.5+1}} = \frac{1}{2s^{0.5+1}}\sqrt{\pi}$.

$L\{t^{-0.5}\} = \frac{\Gamma(-0.5+1)}{s^{-0.5+1}} = \sqrt{\frac{\pi}{s}}$.

## 1.2 Further Properties and Initial-Value Problem

### 1.2.1 LT of the derivative of a function

In the definition of the LT, replace $f(t)$ by it's derivative $f'(t)$:

$$L\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\, dt.$$

Now integrate by parts ($u = e^{-st}$, $dv = f'(t)\, dt$):

$$\int_0^\infty e^{-st} f'(t)\, dt = f(t)e^{-st}|_0^\infty - \int_0^\infty f(t) \cdot (-s) \cdot e^{-st}\, dt = -f(0) + sL\left\{f(t)\right\}.$$

Therefore, if $F(s)$ is the LT of $f(t)$ then $sF(s) - f(0)$ is the LT of $f'(t)$:

$$L\left\{f'(t)\right\} = sL\left\{f(t)\right\} - f(0). \tag{1}$$

Replace $f$ by $f'$ in (1),

$$L\left\{f''(t)\right\} = sL\left\{f'(t)\right\} - f'(0), \tag{2}$$

and apply (1) again:

$$L\left\{f''(t)\right\} = s^2 L\left\{f(t)\right\} - sf(0) - f'(0), \tag{3}$$

This is called a **derivative theorem** for the LT.

By induction, we have

**Theorem 3.** *Suppose $f$ and its derivatives of order up to $n-1$ are continuous and of exponential order with $|f^{(j)}(t)| \le Ke^{at}$ for all $t > M$, $0 \le j \le n-1$, and that $f^{(n)}$ is piece-wise continuous, then $L\{f^{(n)}\}$ exists for $s > a$ and*

$$L\left\{f^{(n)}(t)\right\} = s^n L\left\{f(t)\right\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

Using (1) and (3), the LT of any constant coefficient ODE

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

is

$$a(s^2 L\left\{x(t)\right\} - sx(0) - x'(0)) + b(sL\left\{x(t)\right\} - x(0)) + cL\left\{x(t)\right\} = F(s),$$

where $F(s) = L\left\{f(t)\right\}$. In particular, the LT of the solution, $X(s) = L\left\{x(t)\right\}$, satisfies

$$X(s) = \frac{F(s) + asx(0) + ax'(0) + bx(0)}{as^2 + bs + c}.$$

Note that the denominator is the characteristic polynomial of the DE.

**Example 5.** *Using the Laplace transform solve the differential equation*

$$f''(t) - 4f'(t) + 3f(t) = 1$$

*with boundary conditions* $f(0) = f'(0) = 0$.

**Solution:** Take the Laplace transform of the equation. Since $f'(0) = f(0) = 0$, if $\mathcal{L}(f) = F(s)$ then $\mathcal{L}(f') = sF(s)$ and $\mathcal{L}(f'') = s^2F(s)$. Thus,

$$s^2F - 4sF + 3F = \frac{1}{s}$$

and so

$$\begin{aligned}(s^2 - 4s + 3)F &= \frac{1}{s} \\ F &= \frac{1}{s}\frac{1}{s^2 - 4s + 3} \qquad (4)\end{aligned}$$

and, since $s^2 - 4s + 3 = (s-3)(s-1)$, this gives

$$F = \frac{1}{s(s-3)(s-1)}$$

Before we can invert this, we need to do a partial fraction expansion.

$$\begin{aligned}\frac{1}{s(s-3)(s-1)} &= \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1} \\ 1 &= A(s-3)(s-1) + Bs(s-1) + Cs(s-3) \qquad (5)\end{aligned}$$

So substituting in $s = 0$ we get $A = 1/3$, $s = 3$ gives $B = 1/6$ and $s = 1$ gives $C = -1/2$. Hence

$$F = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}$$

and so

$$f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t}.$$

**Example 6.** *Using the Laplace transform solve the differential equation*

$$f'' - 4f' + 3f = 0$$

*with boundary conditions* $f(0) = 1$ *and* $f'(0) = 1$.

**Solution:** In this example there are non-zero boundary conditions. Since

$$\begin{aligned} \mathcal{L}(f') &= sF - f(0) \quad (6) \\ \mathcal{L}(f'') &= s^2F - sf(0) - f'(0) \quad (7) \end{aligned}$$

the subsidiary equation in this case is

$$s^2F - s - 1 - 4sF + 4 + 3F = 0$$

so

$$(s^2 - 4s + 3)F = s - 3.$$

Hence

$$F = \frac{1}{s-1}$$

and

$$f(t) = e^t$$

### 1.2.2 LT of the integral of a function

**Theorem 4.** *If $f$ is continuous and $L\{f\}$ exists, then*

$$L\left\{\int_0^t f(x)dx\right\} = \frac{L\{f\}}{s}.$$

*Equivalently,*

$$L^{-1}\left\{\frac{L\{f\}}{s}\right\} = \int_0^t f(x)dx.$$

Proof. Let $g(t) = \int_0^t f(x)dx$. Then

$$L\{f(t)\} = L\{g'(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\}.$$

**Example 7.**

$$\begin{aligned} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} &= \frac{\sin at}{a}, \\ L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} &= \int_0^t \frac{\sin ax}{a}dx = \frac{1-\cos at}{a^2}, \\ L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} &= \int_0^t \frac{1-\cos ax}{a^2}dx = \frac{at - \sin at}{a^3}. \end{aligned}$$

**Example 8.** *Solve the integral equation $f(t) + \int_0^t f(x)dx = 1$.*

*Solution:* Taking the Laplace transform we get

$$F(s) + \frac{F(s)}{s} = \frac{1}{s}, \Rightarrow F(s) = \frac{1}{s+1}, \Rightarrow f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

### 1.2.3 The Derivative of LT of a Function

Differentiate the definition of the LT with respect to $s$:

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt.$$

Repeating this:

$$\frac{d^n}{ds^n} F(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt. \tag{8}$$

This gives:

**Theorem 5.** *If $f(t)$ is piecewise continuous and of exponential order, and $L\{f(t)\} = F(s)$, then*

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

*Equivalently,*

$$L^{-1}\{F(s)\} = (-1)^n \frac{L^{-1}\left\{F^{(n)}(s)\right\}}{t^n}.$$

These formulas are used in the following cases:

(i) Find the Laplace transforms of function with the forms $t^n f(t)$, when the transform of $f(t)$ is easy to find.

(ii) Find the inverse transform of $F(s)$ if the inverse transform of the derivative of $F(s)$ is easy to find.

As we know the LT of $f(t) = e^{at}$ is $F(s) = (s-a)^{-1}$. By the theorem we have

**Example 9.**

$$L\{te^{at}\} = -F'(s) = (s-a)^{-2}, \quad L\{t^2 e^{at}\} = F''(s) = 2 \cdot (s-a)^{-3},$$
$$L\{t^3 e^{at}\} = -F'''(s) = 2 \cdot 3 \cdot (s-a)^{-4}, \quad \dots,$$

*and in general*

$$L\left\{t^n e^{at}\right\} = (-1)^n F^{(n)}(s) = n! \cdot (s-a)^{-n-1}. \tag{9}$$

**Example 10.** *Using the Laplace transform solve the differential equation*

$$f'' - 4f' + 3f = 2e^t$$

*with boundary conditions $f(0) = f'(0) = 1$.*

**Solution:** This time we have $L(2e^t) = 2/(s-1)$ on the right hand side. This means that the subsidiary equation is

$$(s^2 - 4s + 3)F = \frac{2}{s-1} + s - 3,$$

so

$$F = \frac{2}{(s-1)^2(s-3)} + \frac{1}{s-1}.$$

We need to do partial fractions again, but this is one of those cases with a repeated root:

$$\frac{1}{(s-1)^2(s-3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-3}$$

and multiplying across

$$1 = A(s-1)(s-3) + B(s-3) + C(s-1)^2$$

so $s = 1$ gives $B = -1/2$ and $s = 3$ gives $C = 1/4$. No value of $s$ gives $A$ on its own, so wee try $s = 2$:

$$1 = -A + \frac{1}{2} + \frac{1}{4}$$

which means that $A = -1/4$. Hence

$$F = \frac{1}{2(s-1)} - \frac{1}{(s-1)^2} + \frac{1}{2(s-3)},$$

and

$$f = \frac{1}{2}e^t - te^t + \frac{1}{2}e^{3t}.$$

**Example 11.** *Using the Laplace transform solve the differential equation*

$$y'' - 2ay' + a^2y = 0$$

*with boundary conditions $y'(0) = 1$ and $y(0) = 0$. $a$ is some real constant.*

**Solution:** Taking the Laplace transform we get

$$s^2Y - 1 - 2aY + a^2Y = 0$$

and hence

$$Y = \frac{1}{(s-a)^2}$$

which means that

$$y = te^{at}$$

**Example 12.** *Using LT solve the DE*

$$x' + x = t^{100}e^{-t}, \quad x(0) = 0.$$

*Note this would be highly impractical to solve using undetermined coefficients. (You would have* 101 *undetermined coefficients to solve for!)*

**Solution:** We apply LT to the two sides of the DE. The LT of the LHS: by (1),

$$L\{x' + x\} = sX(s) + X(s),$$

For the LT of the RHS, let

$$L\left\{e^{-t}\right\} = \frac{1}{s+1}.$$

By (8),

$$L\left\{t^{100}e^{-t}\right\} = \frac{d^{100}}{ds^{100}}\frac{1}{s+1}\frac{100!}{(s+1)^{101}}.$$

Consequently,

$$X(s) = \frac{100!}{(s+1)^{102}}.$$

Using (9), we can compute the ILT of this:

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{\frac{100!}{(s+1)^{102}}\right\} = \frac{1}{101}L^{-1}\left\{\frac{101!}{(s+1)^{102}}\right\} = \frac{1}{101}t^{101}e^{-t}.$$

### 1.2.4 The Integral of LT of a Function

**Theorem 6.** *Let $f(t)$ be piecewise continuous and of exponential order, and $L\{f(t)\} = F(s)$. If $\lim\limits_{t\to 0^+} \frac{f(t)}{t}$ exists, then*

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(x)dx.$$

*Equivalently,*

$$L^{-1}\{F(s)\} = tL^{-1}\left\{\int_s^{+\infty} F(x)dx\right\}.$$

Proof. Let $g(t) = \frac{f(t)}{t}$. Then $f(t) = tg(t)$, $F(s) = L\{f\} = -G'(s)$.

These formulas are used in the following cases:

(i) Find the Laplace transforms of function with the forms $\frac{f(t)}{t}$, when the transform of $f(t)$ is easy to find.

(ii) Find the inverse transform of $F(s)$ if the inverse transform of the integral of $F(s)$ is easy to find.

**Example 13.**

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{+\infty} \frac{a}{x^2+a^2}dx = \frac{\pi}{2} - \arctan(s/a), \quad a > 0.$$

**Example 14.**

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s-1)^3}\right\} &= tL^{-1}\left\{\int_s^{+\infty} \frac{1}{(x-1)^3}dx\right\} \\ &= -t/2L^{-1}\left\{\frac{1}{(s-1)^2}\right\} \\ &= (-t/2)tL^{-1}\left\{\int_s^{+\infty} \frac{1}{(x-1)^2}dx\right\} \\ &= t^2/2L^{-1}\left\{\frac{1}{s-1}\right\} \\ &= \frac{t^2e^t}{2} \end{aligned}$$

### 1.2.5 The First Shifting Theorem

**Theorem 7.** *If F(s) exists for $s > c \geq 0$, then, for any constant $a < s - c$,*

$$L\left\{e^{at}f(t)\right\} = F(s-a).$$

*Equivalently,*

$$L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\}.$$

**Example 15.**

$$L\left\{e^{bt}\sin at\right\} = \frac{a}{(s-b)^2+a^2}.$$

**Example 16.**

$$L^{-1}\left\{\frac{s}{s^2+6s+10}\right\} = L^{-1}\left\{\frac{s}{(s+3)^2+1}\right\}$$

$$= L^{-1}\left\{\frac{s+3-3}{(s+3)^2+1}\right\} = e^{-3t}L^{-1}\left\{\frac{s-3}{s^2+1}\right\} = e^{-3t}(\cos t - 3\sin t).$$

### 1.2.6 The Second Shifting Theorem

Define the unit step (Heaviside) function by

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a, \end{cases}$$

where $a > 0$. In some books, $u(t-a) = H_a(t)$.

**Example 17.**

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt = \int_a^\infty e^{-st}\, dt = \left\{ \frac{e^{-st}}{-s} \right\}_a^\infty = \frac{e^{-as}}{s}, .$$

Assuming $f(t) = 0, t < 0$. The transform of a function $f(t), t \geq 0$, to $u(t-a)f(t-a)$ is called a shift of $f(t)$ by $a$ units.

$$u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{for } t \geq a \\ 0 & \text{for } t < a, \end{cases}$$

**Theorem 8.** *(The Second Shift Theorem) If F(s) exists for* $s > c \geq 0$*, then, for any constant* $a \geq 0$*,*

$$L\{u(t-a)f(t-a)\} = e^{-as} L\{f(t)\}.$$

*Equivalently,*

$$L^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)\left[L^{-1}\{F(s)\}\right]_{t-a}.$$

**Example 18.**

$$L\left\{u(t-3)(t-3)^2\right\} = e^{-3s} L\{t^2\} = \frac{2e^{-3s}}{s^3}.$$

$$L\left\{u(t-3)t^2\right\} = L\left\{u(t-3)[(t-3)^2 + 6(t-3) + 9]\right\} = e^{-3s}[L\{t^2\} + 6L\{t\} + 9L\{1\}]$$
$$= e^{-3s}[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}].$$

If a function is defined as

$$f(t) = \begin{cases} f_1(t), & t < t_1; \\ f_2(t), & t_1 \leq t < t_2; \\ \cdots, & \cdots; \\ f_n(t), & t \geq t_{n-1}. \end{cases}$$ Then

$f(t) = [1 - u(t-t_1)]f_1(t) + [u(t-t_1) - u(t-t_2)]f_2(t) + [u(t-t_2) - u(t-t_3)]f_3(t) + \cdots + [u(t-t_{n-2}) - u(t-t_{n-1})]f_{n-1}(t) + u(t-t_{n-1})f_n(t)$.

**Example 19.** *If* $f(t) = \begin{cases} f_1(t), & t < a; \\ f_2(t), & a \leq t < b; \\ f_3(t), & b \leq t < c; \\ f_4(t), & t \geq c. \end{cases}$ *Then*

$$f(t) = [1-u(t-a)]f_1(t)+[u(t-a)-u(t-b)]f_2(t)+[u(t-b)-u(t-c)]f_3(t)+u(t-c)f_4(t).$$

**Example 20.** *Find* $L\{f(t)\}$*, where* $f(t) = \begin{cases} 0, & t < 0; \\ t, & 0 \leq t < 2; \\ 1, & 2 \leq t < 3; \\ \sin(t), & t \geq 3. \end{cases}$

**Solution:**

$$L\{f(t)\} = L\{[1-u(t)]0 + [u(t)-u(t-2)]t + [u(t-2)-u(t-3)]1 + u(t-3)\sin(t)\}$$

$$= L\{u(t)t - u(t-2)](t-2+1) - u(t-3) + u(t-3)\sin(t-3+3)\}$$

**Example 21.** *Solve the differential equation:*

$$f'' + f' - 6f = \begin{cases} 0 & t < 3 \\ 2 & 3 \leq t < 5 \\ 4 & 5 \leq t \end{cases}$$

*with* $f(0) = f'(0) = 0$.

**Solution:**

$$2u(t-3) - 2u(t-5) + 4u(t-5) = \begin{cases} 0 & t < 3 \\ 2 & 3 \leq t < 5 \\ 4 & 5 \leq t \end{cases}$$

The differential equation is

$$f'' + f' - 6f = 2u(t-3) + 2u(t-5)$$

with $f(0) = f'(0) = 0$ and we take the Laplace transform of both sides:

$$s^2F + sF - 6F = \frac{2}{s}\left(e^{-3s} + e^{-5s}\right)$$

or

$$F(s) = \frac{2}{s(s+3)(s-2)}\left(e^{-3s} + e^{-5s}\right)$$

By partial fractions,

$$\frac{2}{s(s+3)(s-2)} = -\frac{1}{3s} + \frac{2}{15}\frac{1}{s+3} + \frac{1}{5}\frac{1}{s-2} = L\left(-\frac{1}{3} + \frac{2}{15}e^{-3t} + \frac{1}{5}e^{2t}\right)$$

By the Second Shifting Theorem,

$$f(t) = \left(-\frac{1}{3} + \frac{2}{15}e^{-3t+9} + \frac{1}{5}e^{2t-6}\right) u(t-3) + \left(-\frac{1}{3} + \frac{2}{15}e^{-3t+15} + \frac{1}{5}e^{2t-10}\right) u(t-5)$$

**Example 22.** *Solve the differential equation:*

$$f'' + 4f' + 7f = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 4 & 5 \le t \end{cases}$$

*with* $f(0) = f'(0) = 0$.

**Solution:**

$$2u(t-3) - 2u(t-5) + 4u(t-5) = \begin{cases} 0 & t < 3 \\ 2 & 3 \le t < 5 \\ 4 & 5 \le t \end{cases}$$

The differential equation is

$$f'' + 4f' + 7f = 2u(t-3) + 2u(t-5)$$

with $f(0) = f'(0) = 0$ and we take the Laplace transform of both sides:

$$s^2F + 4sF + 7F = \frac{2}{s}\left(e^{-3s} + e^{-5s}\right)$$

or

$$F = \frac{2}{s(s^2+4s+7)}\left(e^{-3s} + e^{-5s}\right)$$

By partial fractions,

$$\frac{2}{s(s^2+4s+7)} = \frac{1}{7s} - \frac{1}{7}\frac{s+2}{(s+2)^2+3} - \frac{1}{7}\frac{2}{(s+2)^2+3}$$

$$= L\left(\frac{1}{7} - \frac{1}{7}e^{-3t}\cos(\sqrt{3}t) - \frac{2}{7\sqrt{3}}e^{-3t}\sin(\sqrt{3}t)\right)$$

The answer will come from the Second Shifting Theorem.

### 1.2.7 Laplace Transform of Periodic Functions

A function $f(t)$ is periodic if there exists $\omega > 0$ such that $f(t+\omega) = f(t)$ for all t. $\omega$ is a period of $f$. The smallest positive period $\omega$ of $f$ is called the fundamental period of $f$, and $f$ is called $\omega$-periodic.

**Theorem 9.** *If $f$ is called $\omega$-periodic for $t \geq 0$, then*

$$L\{f(t)\} = \frac{1}{1-e^{-\omega s}} \int_0^{\omega} e^{-st} f(t)\, dt.$$

Proof.

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = \sum_{n=0}^{\infty} \int_{n\omega}^{(n+1)\omega} e^{-st} f(t)\, dt = \sum_{n=0}^{\infty} \int_0^{\omega} e^{-s(x+n\omega)} f(x)\, dx,$$

where $x = t - n\omega$.

**Example 23.** *If* $f(t) = \begin{cases} t, & 0 \leq t < 1; \\ 0, & 1 \leq t < 2. \end{cases}$ *and 2-periodic, then*

$$L\{f(t)\} = \frac{1-(1+s)e^{-s}}{s^2(1-e^{-2s})}.$$

**Example 24.** *Half-rectified wave:* $f(t) = \begin{cases} \sin t, & 2k\pi \leq t < (2k+1)\pi; \\ 0, & (2k+1)\pi \leq t < (2k+2)\pi. \end{cases}$ *Then*

$$L\{f(t)\} = \frac{1}{(s^2+1)(1-e^{-\pi s})}.$$

**Example 25.** *Full-rectified wave:* $f(t) = \begin{cases} \sin t, & 2k\pi \leq t < (2k+1)\pi; \\ -\sin t, & (2k+1)\pi \leq t < (2k+2)\pi. \end{cases}$ *Then*

$$L\{f(t)\} = \frac{1}{(s^2+1)} coth \frac{\pi s}{2}.$$

## 1.3 Convolutions and generalized Functions

### 1.3.1 Convolutions

The *convolution* of $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t f(t-u)g(u)\,du.$$

Remark. $f * g = g * f$.

**Example 26.** *Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ $(t \geq 0)$.*

**Solution:** From the definition of convolutions

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \tau e^{2(t-\tau)}\,d\tau \\
&= \int_0^t \tau e^{2t}e^{-2\tau}\,d\tau = e^{2t}\int_0^t \tau e^{-2\tau}\,d\tau \\
\text{Use integration by parts} &= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}
\end{aligned}$$

**Convolution theorem**

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s) = \mathcal{L}\{f\}\mathcal{L}\{g\}, \Leftrightarrow \mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

The LT of the convolution is the product of the LTs. (Or, equivalently, the inverse LT of the product is the convolution of the inverse LTs.)

Proof: Do a change-of-variables in the following double integral:

$$\begin{aligned}
\mathcal{L}\{f * g(t)\} &= \int_0^\infty e^{-st}\int_0^t f(u)g(t-u)\,du\,dt \\
&= \int_0^\infty \int_u^\infty e^{-st}f(u)g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su}f(u)\int_u^\infty e^{-s(t-u)}g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su}f(u)\,du\int_0^\infty e^{-sv}g(v)\,dv \\
&= \mathcal{L}\{f\}\mathcal{L}\{g\}.
\end{aligned}$$

**Example 27.** *Calculate the inverse Laplace transform of $\frac{1}{s^3-2s^2}$.*

**Solution:** This can be computed using partial fractions and LT tables. However, it can also be computed using convolutions.

First we factor the denominator, as follows

$$\frac{1}{s^3 - s^2} = \frac{1}{s^2}\frac{1}{s-2}.$$

We know the inverse Laplace transforms of each term:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$$

We apply the convolution theorem:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right] = t * e^{2t} = \int_0^t ue^{2(t-u)}\,du = -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}.$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right](t) = e^t - t - 1.$$

**Example 28.** *Find the convolution*

$$f(t) = 1 * 2 * 3 * 4 * 5.$$

**Solution:** Take the LT. Since the LT of the convolution is the product of the LTs:

$$\mathcal{L}\{1 * 2 * 3 * 4 * 5\} = 5!(1/s)^5 = \frac{5!}{s^5} = F(s).$$

We know from LT tables that $\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = t^4$, so

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = 5\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = 5t^4.$$

### 1.3.2 Dirac delta function

Consider the function

$$\delta_k(t) = \begin{cases} k, & 0 < t \le \frac{1}{k}; \\ 0, & \text{otherwise.} \end{cases} \qquad \delta_k(t-a) = \begin{cases} k, & a < t \le a + \frac{1}{k}; \\ 0, & \text{otherwise.} \end{cases}$$

Properties:

- $\delta_k(t-a) = ku(t-a) - ku(t-(a+\frac{1}{k}))$.
- $\int_{-\infty}^{\infty} \delta_k(t-a)dt = 1$.

- $\int_0^\infty \delta_k(t-a)f(t)dt = \int_a^{a+1/k} kf(t)dt = f(t^*)$, where $t^*$ is a number between $a$ and $a+\frac{1}{k}$.
- $L\{\delta_k(t-a)\} = e^{-as}\frac{k}{s}(1-e^{-s/k})$.
- $\lim\limits_{k\to\infty} L\{\delta_k(t-a)\} = e^{-as}$.

**Definition 1.** *Dirac delta distribution, or Dirac delta (generalized) function, is defined as*

$$\delta(t-a) = \lim_{k\to\infty} \delta_k(t-a).$$

The "Dirac delta function" $\delta(t)$ is technically not a function. It can represent idealized point mass, or point charge.

Properties:

- $L\{\delta(t)\} = 1$.
- $L\{\delta(t-a)\} = e^{-as}$.
- $\delta_a(f) = \int_0^\infty \delta(t-a)f(t)dt = f(a)$, which is a linear functional.
- If $a > 0$ then $\int_0^\infty \delta(t-a)f(t)e^{-st}\,dt = f(a)e^{-as}$.

**Example 29.** *Solve $x'' + x = -\delta(t-\pi)$, $x(0) = x'(0) = 0$.*

Remark. This models a unit mass attached to an undamped spring suspended from a board with no initial displacement or initial velocity. At time $t\pi$, the board is hit very hard (say with a hammer blow) in the upward direction. As we will see, this will start the mass to oscillate.

Solution: Taking Laplace transforms gives $s^2X(s) + X(s) = e^{-\pi s}$, so

$$X(s) = -\frac{1}{s^2+1}e^{-\pi s}.$$

The inverse LT is $x(t) = \sin(t)u(t-\pi)$.

**Example 30.** *Solve*

$$f'' + f' - 6f = \delta(t-4)$$

*with* $f(0) = f'(0) = 0$.

**Solution:** Using $\mathcal{L}[\delta(t-a)] = e^{-as}$ this gives

$$s^2F + sF - 6F = e^{-4s}$$

or

$$F = \frac{e^{-4s}}{(s+3)(s-2)}$$

By partial fractions we have

$$\frac{1}{(s+3)(s-2)} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} = \mathcal{L}\left(\frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}\right).$$

By the Second Shifting Theorem,

$$f(t) = \left(\frac{1}{5}e^{2(t-4)} - \frac{1}{5}e^{-3(t-4)}\right) u(t-4).$$

# 2 Series Solutions of ODEs

## 2.1 Basic Concepts

We have fully investigated solving second order linear differential equations with constant coefficients:

$$Ay'' + By' + Cy = 0,$$

where A,B,C are constants. Now we will explore how to find solutions to second order linear differential equations whose coefficients are not necessarily constant:

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

**Definition 2.** *The Taylor series about $x_0$ of a function $f(x)$ is the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

*There exists $R \geq 0$ such that the series is convergent in $|x - x_0| < R$ and divergent in $|x - x_0| > R$. The number $R$ is called **Radius of Convergence**. We have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \quad |x - x_0| < R.$$

Remark. We use Ratio Test to find $R$.

**Example 31.** $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,\ R = 1.$

$\frac{1}{3-2x} = \frac{1}{1-2(x-1)} = \sum_{n=0}^{\infty} 2^n(x-1)^n.\ R = \frac{1}{2}.$

$\frac{1}{3-2x} = \frac{1}{3}\frac{1}{1-\frac{2}{3}x} = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{2}{3})^n x^n.\ R = \frac{3}{2}.$

**Theorem 10.** *If*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$$

*for all $x$ in an interval $I$, then $a_n = 0, n \geq 0$.*

### 2.1.1 Analytic functions

**Definition 3.** *A function $f(x)$ is analytic at $x_0$ if $f$ has Taylor series about $x_0$ which converges to $f(x)$ in an interval containing $x_0$.*

**Example 32.** *$f(x) = \frac{1}{1-x}$ is analytic at $x = 0$; $f(x) = \sqrt{x}$ is not analytic at 0, since $f'(0)$ does not exist; $f(x) = e^x$ is analytic at any $x$.*

Remark: If $f$ and $g$ are analytic at $x_0$, then $cf$, $f \pm g$, $fg$, $f/g$ (if $g(x_0) \neq 0$) are analytic at $x_0$.
Remark: If $f$ is analytic at $x_0$, then its Taylor series about $x_0$ is unique.

### 2.1.2 Singular point and ordinary point

Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If we divide two sides by $P(x)$, then the equation is changed to

$$y'' + p(x)y' + q(x)y = 0. \qquad (1)$$

**Definition 4.** *If both $p(x)$ and $q(x)$ are analytic at a point $x_0$, then $x_0$ is called an ordinary point. Otherwise, it is called a singular point.*

Remark. If $p(x)$ and $q(x)$ are polynomials, then any point is an ordinary point.

**Example 33.** *The following equation has singular points $x = 1, 2$:*

$$y'' + \frac{x + e^x}{x - 1}y' + \frac{x + 1}{x - 2}y = 0.$$

**Theorem 11.** *If $x_0$ is an ordinary point of the ODE (1), then the general solution of (1) is analytic at $x_0$, and is therefore given by*

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

*in an open interval containing $x_0$ with two arbitrary coefficients (usually $a_0$ and $a_1$). The radius $R$ is at least the distance from $x_0$ to the closest singular point of the equation.*

**Example 34.** *Consider the following equation with two singular points $x = 1, 2$:*

$$y'' + \frac{x + e^x}{x - 1}y' + \frac{x + 1}{x - 2}y = 0.$$

*Then the series solution about $x_0 = 0$ has $R \geq 1$; the series solution about $x_0 = \frac{3}{4}$ has $R \geq \frac{1}{4}$.*

**Example 35.** *Consider the following equation:*

$$y'' + \frac{1}{x^2+1}y' + xy = 0.$$

*Then the series solution about $x_0 = 0$ has $R \geq 1$; the series solution about $x_0 = 2$ has $R \geq \sqrt{5}$.*

**Definition 5.** *Let $x_0$ be a singular point. If both $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at $x_0$, then $x_0$ is called a* ***regular singular point*** *(non-essential singular point). If at least one of $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ is not analytic at $x_0$, then $x_0$ is called an* ***irregular singular point*** *(essential singular point).*

**Example 36.** *Find all singularities and classify them:*
*(1)* $y'' + \frac{x-1}{x(x-2)}y' + \frac{x}{x^2(x-2)^3}y = 0.$
*(2)* $(x\sin x)y'' + (\cos x)y' + e^x y = 0.$

**Solution:** (1) It has one regular singular point $x = 0$ and one irregular singular point $x = 2$.

(2) We change the equation to

$$y'' + \frac{\cos x}{x\sin x}y' + \frac{e^x}{x\sin x}y = 0.$$

Note that $\frac{\cos x}{x\sin x}$ and $\frac{\cos x}{x\sin x}$ are not analytic at zeros of the denominator $x\sin x$. Let $x\sin x = 0$, we imply that $x = k\pi$, $k = 0, \pm 1, \pm 2, ....$They are all singular points, in which $x = 0$ is irregular, and others are regular.

## 2.2 Solutions About Ordinary Points

Now we solve an equation at ordinary points.

**Example 37.** *Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$ be the series solution about $x = 0$ of the differential equation: $(x^2+1)y'' - 4xy' + 6y = 0$. Find the recursive relation to the coefficients $a_n$, and then solve the equation.*

**Solution:** From $y = \sum_{n=0}^{\infty} a_n x^n$ we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Substitute all of them into the DE,

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} na_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0. \Rightarrow$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 4na_n x^n + 6\sum_{n=0}^{\infty} 6a_n x^n = 0. \Rightarrow$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0. \Rightarrow$$

$$\sum_{n=0}^{\infty}[n(n-1)a_n - 4na_n + 6a_n + (n+2)(n+1)a_{n+2}]x^n = 0. \Rightarrow$$

$$n(n-1)a_n - 4na_n + 6a_n + (n+2)(n+1)a_{n+2} = 0, \Rightarrow$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n.$$

We imply that

$$a_2 = -3a_0, \quad a_3 = -\frac{1}{3}a_1, \quad a_4 = a_5 = ... = 0.$$

Thus

$$y = a_0\left(1-3x^2\right) + a_1\left(x - \frac{1}{3}x^3\right).$$

**Example 38.** *Consider the DE: $y'' - 2xy' + y = 0$. Note that $x_0 = 0$ is a regular point. So we shall attempt to find a series solution in the form:*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

*1) Find the recursive relation of the coefficients in the series solution about $x_0 = 0$.*
*2) Solve the recursive relation.*
*3) Find the particular solution with $y(0) = 1, y'(0) = 2$.*

**Solution:** 1) From $y = \sum_{n=0}^{\infty} a_n x^n$ we have

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}, \Rightarrow xy' = \sum_{n=1}^{\infty} na_n x^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n.$$

Substitute all of them into the DE, we imply that

$$a_{n+2} = \frac{2n-1}{(n+1)(n+2)} a_n.$$

2) From the recursive relation above,

$$a_{2n} = \frac{3 \cdots 7 \cdot 11 \cdots (4n-5)}{(2n)!} a_0,$$

$$a_{2n+1} = \frac{1 \cdots 5 \cdot 9 \cdots (4n-3)}{(2n+1)!} a_1.$$

We have

$$y = a_0 \left( 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdots 7 \cdot 11 \cdots (4n-5)}{(2n)!} x^{2n} \right)$$

$$+a_1 \left( x + \sum_{n=1}^{\infty} \frac{1 \cdots 5 \cdot 9 \cdots (4n-3)}{(2n+1)!} x^{2n+1} \right)$$

3) From $y(0) = 1$ we have $a_0 = 1$; by $y'(0) = 2$, we get $a_1 = 2$. Thus

$$y = \left( 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdots 7 \cdot 11 \cdots (4n-5)}{(2n)!} x^{2n} \right)$$

$$+2 \left( x + \sum_{n=1}^{\infty} \frac{1 \cdots 5 \cdot 9 \cdots (4n-3)}{(2n+1)!} x^{2n+1} \right)$$

**Example 39.** *Consider the DE: $y'' - 2xy' + sy = 0$, about $x_0 = 0$, where $s$ is a constant. Let the series solution be in the form:*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

*Then the recursive relation of the coefficients is*

$$a_{2n+2} = \frac{(-1)^{n+1} s(s-4)(s-8)...(s-4n)}{(2n+2)!} a_0, \quad a_{2n+3} = \frac{(-1)^{n+1}(s-2)(s-6)...(s-4n-2)}{(2n+3)!} a_1.$$

## 2.3 Solutions About Regular Singular Points

### 2.3.1 Cauchy-Euler Equations

An equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + ... + a_1 x y' + a_0 y = 0,$$

is called a *Cauchy-Euler* equation.

**Theorem 12.** *Second-order Cauchy-Euler equation:*

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0, \quad x \neq 0,$$

*with* $a_2$, $a_1$ *and* $a_0$ *constants. In standard form, the equation is*

$$y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0, \quad x \neq 0.$$

*The indicial equation (or, auxiliary equation) is*

$$a_2 r(r-1) + a_1 r + a_0 = 0, \quad or, \quad r^2 + (A-1) r + B = 0.$$

*Let* $r_1$ *and* $r_2$ *be the two solutions of the indicial equation.*

*(i) If* $r_1 \neq r_2$ *are real, then* $y_1 = |x|^{r_1}$ *and* $y_2 = |x|^{r_2}$.

*(ii) If* $r_1 = r_2$ *(real), then* $y_1 = |x|^{r_1}$ *and* $y_2 = x^{r_1} \ln|x|$.

*(iii) If* $r_1, r_2 = \alpha \pm i\beta$ *(complex), then* $y_1 = |x|^{\alpha} \cos(\beta \ln|x|)$ *and* $y_2 = |x|^{\alpha} \sin(\beta \ln|x|)$.

Proof. Since $p(x) = \dfrac{A}{x}$ and $q(x) = \dfrac{B}{x^2}$ are undefined at $x = 0$, the solution may be undefined at $x = 0$. Thus, we assume that $x \neq 0$. A Cauchy-Euler equation can be transformed into a constant-coefficient equation as follows:

For $x > 0$, let $x = e^t$ and $y(x) = z(t)$. Then $t = \ln(x)$ and, by the chain rule,

$$\frac{dy}{dx} = \frac{dz}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dz}{dt}, \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dz}{dt} + \frac{1}{x}\frac{d^2z}{dt^2}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dz}{dt} + \frac{1}{x^2}\frac{d^2z}{dt^2},$$

and the equation $x^2 y'' + A x y' + B y = 0$ becomes $\left[\dfrac{d^2z}{dt^2} - \dfrac{dz}{dt}\right] + A\dfrac{dz}{dt} + Bz = 0$, or

$$z'' + (A-1) z' + Bz = 0,$$

which has constant coefficients.

If $z_1(t)$ and $z_2(t)$ are two independent solutions of $z'' + (A-1)z' + Bz = 0$, then two independent solutions of $x^2y'' + Axy' + By = 0$ are given by

$$y_1(x) = z_1(\ln x) \text{ and } y_2(x) = z_2(\ln x).$$

Since solutions of a constant-coefficient equation are sought in the form $z = e^{rt}$ and $y(x) = z(t)$ with $t = \ln(x)$, $y(x) = e^{rt} = e^{r\ln(x)} = e^{\ln(x^r)} = x^r$. Thus, solutions of an Euler equation can be sought directly in the form $y = x^r$.

If $r_1 \neq r_2$ are real, then $z_1 = e^{r_1 t}$ and $z_2 = e^{r_2 t} \to y_1 = x^{r_1}$ and $y_2 = x^{r_2}$.

If $r_1 = r_2$ (real), then $z_1 = e^{r_1 t}$ and $z_2 = te^{r_1 t} \to y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln(x)$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $z_1 = e^{\alpha t}\cos(\beta t)$ and $z_2 = e^{\alpha t}\sin(\beta t) \to y_1 = x^\alpha \cos[\beta\ln(x)]$ and $y_2 = x^\alpha \sin[\beta \ln(x)]$.

For $x < 0$, let $x = -e^t$ and $y(x) = z(t)$. Then $t = \ln(-x)$, and the same equation for $z(t)$ results. In either case, $t = \ln|x|$.

Since $|x| = \left\{ \begin{array}{rl} x, & x > 0 \\ -x, & x < 0 \end{array} \right\}$, replacing $x$ by $|x|$ gives the solutions for any $x \neq 0$. Thus,

If $r_1 \neq r_2$ are real, then $y_1 = |x|^{r_1}$ and $y_2 = |x|^{r_2}$.

If $r_1 = r_2$ (real), then $y_1 = |x|^{r_1}$ and $y_2 = x^{r_1}\ln|x|$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = |x|^\alpha \cos(\beta\ln|x|)$ and $y_2 = |x|^\alpha \sin(\beta \ln|x|)$.

**Example 40.** *Solve the following equations:*

1. $x^2y'' + 2xy' - 2y = 0,\ x > 0.$
   **Solution:** This is an Euler equation. The indicial equation is
   $$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow y = c_1x^{-2} + c_2x = \frac{c_1}{x^2} + c_2x.$$
2. $x^2y'' + 5xy' + 4y = 0,\ x > 0.$

**Solution:** This is an Euler equation. The indicial equation is

$$r^2 + 4r + 4 = 0 \;\Rightarrow\; (r+2)^2 = 0 \;\Rightarrow\; y = \frac{c_1}{x^2} + c_2 \frac{\ln(x)}{x^2}.$$

3. $x^2 y'' + 4xy' + 4y = 0,\ x > 0.$

**Solution:** This is an Euler equation. The indicial equation is

$$r^2 + 3r + 4 = 0 \;\Rightarrow\; r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i \;\Rightarrow$$

$$y = x^{-\frac{3}{2}} \left[ c_1 \cos\left( \frac{\sqrt{7}}{2} \ln(x) \right) + c_2 \sin\left( \frac{\sqrt{7}}{2} \ln(x) \right) \right].$$

4. $3(x-5)^2 y'' + 6(x-5)y' - 6y = 0,\ x \neq 5.$

**Solution:** This is an Euler equation. The indicial equation is

$$r^2 + r - 2 = 0 \;\Rightarrow\; (r+2)(r-1) = 0 \;\Rightarrow y = c_1|x-5|^{-2} + c_2|x-5|.$$

### 2.3.2 The Frobenius Method

Frobenius method is important for equations with coefficients that have singularities, so that power series method can no longer handle them.

**Theorem 6.** Let $x_0 = 0$ be a regular singular point of the DE

$$y'' + p(x)y' + q(x)y = 0$$

with

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let

$$y = x^r \sum_{n=0}^{\infty} c_n(r) x^n = \sum_{n=0}^{\infty} c_n(r) x^{n+r}, \quad c_0(r) = 1.$$

Let $r_1$ and $r_2$ (where $r_1 \geq r_2$ if both are real) be two roots of the indicial equation

$$r^2 + (p_0 - 1)r + q_0 = 0.$$

Then near $x_0 = 0$ for $x > 0$,

$$y_1 = \sum_{n=0}^{\infty} c_n(r_1) x^{n+r_1}.$$

To find a second linearly independent solution $y_2$, we consider three cases:

Case (i): If $r_1 - r_2$ is not an integer, then the two linearly independent solutions are given by:

$$y_2 = \sum_{n=0}^{\infty} c_n(r_2) x^{n+r_2}.$$

Case (ii): If $r_1 = r_2$, then the two linearly independent solutions are given by:

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_1},$$

where $b_n = c_n'(r_1)$.

Case (iii): If $r_1 - r_2 = N$ is a positive integer, then the two linearly independent solutions are given by:

$$y_2 = A y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2},$$

where

$$A = \lim_{r \to r_2} [(r - r_2) c_N(r)], \quad b_n = \frac{d}{dr} [(r - r_2) c_n(r)]_{r=r_2}.$$

The radius of convergence of every one of the above is at least as great as the distance from $x_0 = 0$ to the nearest other singular point of the equation.

**Example 41.** *Find the general solution of*

$$2x^2y'' + (x + 2x^2)y' - y = 0$$

*for* $x > 0$ *near* $x_0 = 0$.

**Solution:** Step 1: Determine whether $x_0 = 0$ is an ordinary point or a regular singular point. Write the equation as

$$y'' + (\frac{1}{2x} + 1)y' - \frac{1}{2x^2}y = 0,$$

We have

$$xp(x) = \frac{1}{2} + x, x^2q(x) = -\frac{1}{2}.$$

So $x_0 = 0$ is a regular singular point.

Step 2: Find and solve the indicial equation. Note that $p_0 = \frac{1}{2}, q_0 = -\frac{1}{2}$. The indicial equation is:

$$r^2 + (p_0 - 1)r + q_0 = 0. \Rightarrow r^2 - \frac{1}{2}r - \frac{1}{2} = 0. \Rightarrow r_1 = 1, r_2 = -\frac{1}{2}.$$

Note that $r_1 - r_2 = 1.5$, so we have Case (i).

Step 3: Find the recursive relation about $c_n(r)$. Let

$$y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}, \quad c_0(r) = 1. \Rightarrow$$

$$y' = \sum_{n=0}^{\infty}(n + r)c_n(r)x^{n+r-1}, y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n(r)x^{n+r-2}.$$

Substitute them into (5.1) we have

$$2x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2} + (x+2x^2)\sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1} - \sum_{n=0}^{\infty} c_n(r)x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty} 2(n + r)(n + r - 1)c_n(r)x^{n+r} + \sum_{n=0}^{\infty}(n + r)c_n(r)x^{n+r}$$

$$+ \sum_{n=0}^{\infty} 2(n + r)c_n(r)x^{n+r+1} - \sum_{n=0}^{\infty} c_n(r)x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)+(n+r)-1]c_n(r)x^{n+r}+\sum_{n=0}^{\infty}2(n+r)c_n(r)x^{n+r+1}=0,\Rightarrow$$

$$\sum_{n=0}^{\infty}[(2n+2r+1)(n+r-1)]c_n(r)x^{n+r}+\sum_{n=1}^{\infty}2(n-1+r)c_{n-1}(r)x^{n+r}=0,\Rightarrow$$

$$(2r+1)(r-1)]c_0(r)x^r+\sum_{n=1}^{\infty}\{[(2n+2r+1)(n+r-1)]c_n(r)+2(n-1+r)c_{n-1}(r)\}x^{n+r}=0,\Rightarrow$$

$$(2r+1)(r-1)]c_0(r)=0, \quad \{[(2n+2r+1)(n+r-1)]c_n(r)+2(n-1+r)c_{n-1}(r)\}=0, n\geq 1.$$

Since $c_0(r)=1\neq 0$, the above first equation results in our indicial equation $(2r+1)(r-1)=0$. The second equation gives

$$c_n(r)=\frac{-2}{2n+2r+1}c_{n-1}(r), \quad n\geq 1. \tag{10}$$

Step 4: Find $y_1$. Take $r=r_1=1$, by (10),

$$c_n(1)=\frac{-2}{2n+3}c_{n-1}(1), n\geq 1.\Rightarrow$$

$$c_1(1)=\frac{-2}{5}c_0(1)=\frac{-2}{5},$$

$$c_2(1)=\frac{-2}{7}c_1(1)=\frac{(-2)^2}{5(7)},...,$$

$$c_n(1)=\frac{(-2)^n}{5(7)...(2n+3)}, n\geq 1.$$

Therefore,

$$y_1=x+\sum_{n=1}^{\infty}\frac{(-2)^n}{5(7)...(2n+3)}x^{n+1}=x\left(1+\sum_{n=1}^{\infty}\frac{(-2)^n}{5(7)...(2n+3)}x^n\right).$$

Step 5: Find $y_2$. Take $r=r_2=-\frac{1}{2}$, by (10),

$$c_n(-\frac{1}{2})=\frac{-2}{2n}c_{n-1}(-\frac{1}{2})=\frac{-1}{n}c_{n-1}(-\frac{1}{2}), n\geq 1.\Rightarrow$$

$$c_1(-\frac{1}{2})=\frac{-1}{1}c_0(-\frac{1}{2})=-1,$$

$$c_2(-\frac{1}{2})=\frac{-1}{2}c_1(-\frac{1}{2})=\frac{(-2)^2}{2},...,$$

$$c_n(-\frac{1}{2})=\frac{(-1)^n}{n!}, n\geq 1.$$

Therefore,

$$y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n-\frac{1}{2}} = x^{-\frac{1}{2}} e^{-x}.$$

Step 6: The general solution is $y(x) = c_1 y_1 + c_2 y_2$, where $c_1$ and $c_2$ are constants.

Remark. To get solutions for $x < 0$ or for $x \neq 0$ near $x_0 = 0$, replace $x^{r_1}$, $x^{r_2}$ and $\ln x$ in Theorem 6 by $|x|^{r_1}$, $|x|^{r_2}$ and $\ln |x|$.

Remark. If $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are not polynomials but are given by Taylor series, then it's possible to obtain approximate solutions.

**Example 42.** *Solve Cauchy-Euler equation*

$$x^2 y'' + 2xy' + \frac{3}{16} y = 0, \quad x > 0$$

Solution. The indicial equation is:

$$r^2 + \frac{1}{2} r - \frac{3}{16} = 0.$$

The two solutions are $r_1 = \frac{1}{4}$, $r_2 = -\frac{3}{4}$, $r_1 - r_2 = 1$.

Substitute

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n(r) x^{n+r}, \quad c_0(r) = 1 \\ y' &= \sum_{n=0}^{\infty} (n+r) c_n(r) x^{n+r-1}, \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n(r) x^{n+r-2} \end{aligned}$$

into the equation, we have

$$\sum_{n=0}^{\infty} (n + r - \frac{1}{4})(n + r + \frac{3}{4}) c_n(r) x^{n+r} = 0, \Rightarrow$$

$$c_n(r_1) = c(r_2) = 0, \quad n \geq 1.$$

We get a basis $y_1 = x^{-1/4}$, $y_2 = x^{-3/4}$.

**Example 43.** *Solve Hyper-geometric differential equation*

$$x(x-1)y'' + (3x-1)y' + y = 0, \quad x > 0. \tag{11}$$

Solution. Note that $p(x) = (3x-1)/(x-1)$ and $q(x) = x/(x-1)$. We have $b_0 = b(0) = 1$, $c_0 = c(0) = 0$. Thus the indicial equation is $r^2 = 0$, which has double root $r = 0$. We have Case (ii). Substitute

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n(r)x^{n+r}, \quad c_0(r) = 1 \\ y' &= \sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1}, \\ y'' &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2} \end{aligned}$$

into (11), we have

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r}$$

$$-\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0,$$

which implies that

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)c_n - (n+r+1)(n+r)c_{n+1} + 3(n+r)c_n - (n+r+1)c_{n+1} + c_n]x^{n+r} - r^2 c_0 x^{r-1} = 0.$$

Simplifying this and using indicial equation $r^2 = 0$, we get

$$\sum_{n=0}^{\infty}(n+r+1)^2(c_n - c_{n+1})x^{n+r} = 0.$$

Thus $c_{n+1} = c_n = \cdots = 1$. So

$$y_1(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

To get another independent solution, note that $b_n = c_n'(r_1) = 0$ for all $n$,

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} = \frac{\ln x}{1-x}.$$

**Example 44.** *(Case III) Solve*

$$(x^2 - x)y'' - xy' + y = 0, \quad x > 0. \tag{12}$$

Solution: The indicial equation is $r(r-1)=0$, the roots are $r_1=1$ and $r_2=0$. This is the Case III with $N=r_1-r_2=1$.

Let $y=\sum_{n=0}^{\infty}c_n(r)x^{n+r}$. Substitute this into (12),

$$(x^2-x)\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1}+\sum_{n=0}^{\infty}c_n(r)x^{n+r}=0.$$

Collecting like terms

$$-r(r-1)x^{r-1}+\sum_{n=0}^{\infty}\left[(n+r-1)^2c_n(r)-(n+r)(n+r+1)c_{n+1}(r)\right]x^{n+r}=0.$$

We derive

$$c_{n+1}(r)=\frac{(n+r-1)^2}{(n+r)(n+r+1)}c_n(r)\quad (n=0,1,...),\Rightarrow$$

When $r=r_1=1$, $c_1=c_2=\cdots=0$. We get a solution $y_1(x)=c_0x=x$.

To get a second solution, we solve $c_n(r)$:

$$c_1(r)=\frac{(r-1)^2}{(r)(r+1)},\quad c_2(r)=\frac{(r)^2}{(r+1)(r+2)}c_1(r)=\frac{(r-1)^2(r)}{(r+1)^2(r+2)},$$

$$c_n(r)=\frac{(r-1)^2(r)}{(r+n-1)^2(r+n)},\quad n\geq 2.$$

$$A=\lim_{r\to r_2}[(r-r_2)c_N(r)]=\lim_{r\to 0}[(r-0)c_1(r)]=1,$$

$$b_n=\frac{d}{dr}[(r-r_2)c_n(r)]_{r=r_2},\Rightarrow b_0=1,b_n=0,b_1=-3,n\geq 2.$$

Thus

$$y_2=Ay_1\ln(x)+\sum_{n=0}^{\infty}b_nx^{n+r_2}=x\ln x+1.$$

Remark. Another method to get a second solution. We let

$$y_2=ky_1\ln(x)+\sum_{n=0}^{\infty}d_nx^{n+r_2}=kx\ln(x)+\sum_{n=0}^{\infty}d_nx^n,$$

where $k$ is a constant. Substitute $y_2$ into the differential equation,

$$-k+\sum_{n=0}^{\infty}[d_n(n-1)^2-d_{n+1}(n+1)n]x^n=0,\Rightarrow$$

$$d_0=k,d_2=d_3=...=0.$$

Thus

$$y_2=kx\ln(x)+k+d_1x.$$

### 2.3.3 Bessel's Equation

**Definition 6.** *The DE*

$$x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0$$

*is called Bessel's equation of order $\nu$ with parameter $\lambda$, where $\nu \geq 0$ and $\lambda$ are constants. If $\lambda = 1$, then it is called Bessel's equation of order $\nu$.*

If $\lambda = 0$, it is Euler equation. If $\lambda \neq 0$, by substitution $z = \lambda x$ and $w(z) = y(x)$, the equation above can be changed to

$$z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0.$$

**Definition 7.**

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

*is called the Bessel function of the first kind of order $\nu$; For any $\nu \geq 0$,*

$$Y_\nu(x) = \lim_{\mu \to \nu} \frac{\cos(\pi\mu)J_\mu(x) - J_{-\mu}(x)}{\sin(\pi\mu)}$$

*is called the Bessel function of the second kind of order $\nu$.*

**Theorem** For any $x > 0$, two linearly independent solutions $y_1(x)$ and $y_2(x)$ of

$$x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0$$

are

$$y_1(x) = J_\nu(\lambda x), \quad y_2(x) = \begin{cases} J_{-\nu}(\lambda x), & \text{if } \nu > 0 \text{ is not an integer;} \\ Y_\nu(\lambda x), & \text{for any } \nu \geq 0. \end{cases}$$

The general solution is $y(x) = c_1y_1 + c_2y_2$, where $c_1$ and $c_2$ are constants.

Proof. Note that $x = 0$ is a regular singular point. We can use Frobenious method. The indicial equation is

$r^2 - \nu^2 = 0 \Rightarrow r_1 = \nu, r_2 = -\nu.$

Let $y = \sum_{n=0}^{\infty} c_nx^{n+\nu}$. Substitute this into the DE,

$$c_1(1 + 2\nu)x^{1+\nu} + \sum_{n=2}^{\infty}[n(n + 2\nu)c_n + c_{n-2}\lambda^2]x^{n+\nu} = 0.$$

We derive

$$c_1 = 0, \quad c_n = -\frac{\lambda^2}{n(n + 2\nu)}c_{n-2} \quad (n = 0, 1, ...), \Rightarrow$$

$$c_1 = c_3 = \cdots = 0, \quad c_{2n} = (-1)^n \frac{\lambda^{2n}}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)} c_0 \quad (n = 0, 1, ...).$$

Note that

$$\Gamma(n+\nu+1) = (1+\nu)(2+\nu)\cdots(n+\nu)\Gamma(1+\nu).$$

We have

$$c_{2n} = (-1)^n \frac{\lambda^{2n+\nu}\lambda^{-\nu}2^\nu\Gamma(1+\nu)c_0}{2^{2n+\nu}n!\Gamma(n+\nu+1)} \quad (n = 0, 1, ...).$$

By taking $c_0$ such that $\lambda^{-\nu}2^\nu\Gamma(1+\nu)c_0 = 1$, we get the result.

**Remark.** If $\nu > 0$ is an integer, then $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

**Example 45.** *Find two linearly independent solutions, valid for $x > 0$:*

*(1)*

$$x^2y''(x) + xy'(x) + (4x^2 - 9)y(x) = 0.$$

***Solution:*** *Note that $\lambda^2 = 4$ and $\nu^2 = 9$, $\Rightarrow \lambda = 2$ and $\nu = 3$. Hence*

$$y_1(x) = J_3(2x), \quad y_2(x) = Y_3(2x).$$

*(2)*

$$x^2y''(x) + xy'(x) + (4x^2 - 0.25)y(x) = 0.$$

***Solution:*** *Note that $\lambda^2 = 4$ and $\nu^2 = 0.25$, $\Rightarrow \lambda = 2$ and $\nu = 0.5$. Since $\nu$ is not an integer,*

$$y_1(x) = J_{0.5}(2x), \quad y_2(x) = J_{-0.5}(2x).$$

*(3)*

$$xy''(x) + y'(x) + 0.04xy(x) = 0.$$

***Solution:*** *This is not Bessel equation. However, by multiplying two sides with $x$, we get*

$$x^2y''(x) + xy'(x) + 0.04x^2y(x) = 0$$

*which is Bessel equation with $\lambda^2 = 0.04$ and $\nu^2 = 0$, $\Rightarrow \lambda = 0.2$ and $\nu = 0$. Since $\nu$ is an integer,*

$$y_1(x) = J_0(0.2x), \quad y_2(x) = Y_0(0.2x).$$

Remark. If we need solution for $x \neq 0$, then

$$J_\nu(x) = \left(\frac{|x|}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}$$

# 3 Fourier Series

## Pre-knowledge

1. Trig Identities:

$$\sin(n\pi + \frac{\pi}{2}) = (-1)^n; \quad \cos(n\pi) = (-1)^n, \quad n \text{ is an integer.}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}.$$

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}, \quad \cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}.$$

$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}.$$

2. A function $f(x)$ is called $p-$*periodic* if $p > 0$ is the smallest number such that $f(x+p) = f(x)$ for any $x$. The number $p$ is called the period. For example, $\cos kx$ and $\sin kx$ are $\frac{2\pi}{k}$-periodic.

3. Odd-Even function: If $f(-x) = -f(x)$ for all $x \in [-a, a]$, then $f(x)$ is odd on $[-a, a]$; If $f(-x) = f(x)$ for all $x \in [-a, a]$, then $f(x)$ is even on $[-a, a]$. For example, $\sin kx$ is odd, $\cos kx$ is even.

   - If $f(x)$ is odd on $[-a, a]$, then $\int_{-a}^{a} f(x)dx = 0$.
   - If $f(x)$ is even on $[-a, a]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

4. If $m$, $n$ are non-negative integers, then

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0. \quad m \neq n$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0.$$

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi, \qquad n \geq 1.$$

5. Integration by parts: $\int_a^b u dv = uv|_a^b - \int_a^b v du$.

6. $\int (ax+b)\sin(kx)dx = -(ax+b)\frac{1}{k}\cos(kx) + \frac{a}{k^2}\sin(kx) + C$;

   $\int (ax+b)\cos(kx)dx = (ax+b)\frac{1}{k}\sin(kx) + \frac{a}{k^2}\cos(kx) + C$.

7. Integration by substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

## 3.1 Periodic Functions

Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series. Fourier series have many applications such as, solving partial differential equations, signal precessing, image processing.

A function $f(x)$ is **piecewise continuous** in interval (a, b) if we have $a = t_0 < t_1 < ... < t_m = b$, such that f(x) is continuous in each interval $(t_i, t_{i+1})$ and the limits $\lim_{x \to t_i^-} f(x)$ and $\lim_{x \to t_i^+} f(x)$ exist for all $i = 0, 1, 2, \ldots, m$. In the following, we assume that both $f$ and $f'$ are piecewise continuous.

**Definition**. Let $f(x)$ be $2L$-periodic function. Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\}. \tag{13}$$

This series is called the (full) Fourier series for $f(x)$. The coefficients $a_n$ $(n \geq 0)$ are called the Fourier cosine coefficients, and the coefficients $b_n$ $(n \geq 1)$ are called the Fourier sine coefficients.

**Remark**. The "=" occurs at every $x \in [-L, L]$ where $f$ is continuous. If we omit the condition where $f$ is continuous at $x$, then we may write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})).$$

**Theorem**. The Fourier coefficients can be calculated as follows:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \qquad n = 0, 1, 2, \ldots. \tag{14}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx \qquad n = 1, 2, 3, \ldots. \tag{15}$$

Proof. The coefficient $a_0$ is the simplest to find: integrating (22) from $-L$ to $L$,

$$\begin{aligned} \int_{-L}^{L} f(x)\, dx &= \int_{-L}^{L} \frac{a_0}{2}\, dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-L}^{L} \cos(\frac{n\pi x}{L})\, dx + a_n \int_{-L}^{L} \sin(\frac{n\pi x}{L})\, dx \right\} \\ &= \int_{-L}^{L} \frac{a_0}{2}\, dx \Rightarrow \end{aligned}$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx.$$

We do the same thing to compute, say, $b_m$, except that first we multiply (22) through by $\sin(\frac{m\pi x}{L})$. We get

$$\begin{aligned}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx &= \int_{-L}^{L}\frac{a_0}{2}\sin(\frac{m\pi x}{L})\,dx + \\ &\sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx + b_n\int_{-L}^{L}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx\right\} \\ &= b_m\int_{-L}^{L}\sin(\frac{m\pi x}{L})\sin(\frac{m\pi x}{L})\,dx = b_m L. \Rightarrow\end{aligned}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx \qquad m = 1, 2, 3, \ldots.$$

Likewise we can get the formula for $a_m$.

Formulas (14) and (15) allow us to compute the Fourier coefficients of $f$.

**Remark 1**. Even though $f$ is defined only on $[-L, L]$, the right-hand side of (22) is $2L$-periodic, so we could view $f$ as being defined over the whole line, but $2L$-periodic as well.

**Remark 2**. If $f$ is even on $[-L, L]$, then $f(x)\sin(\frac{m\pi x}{L})$ is odd on $[-L, L]$, so $b_n = 0$ for all $n \geq 1$; and $f(x)\cos(\frac{m\pi x}{L})$ is even on $[-L, L]$, so

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx \qquad n = 0, 1, 2, \ldots.$$

**Remark 3**. If $f$ is odd on $[-L, L]$, then $f(x)\cos(\frac{m\pi x}{L})$ is odd on $[-L, L]$, so $a_n = 0$ for all $n \geq 0$; and $f(x)\sin(\frac{m\pi x}{L})$ is even on $[-L, L]$, so

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx \qquad n = 1, 1, 2, \ldots.$$

**$2\pi$-periodic function:**

If $f$ is $2\pi$-periodic (i.e., $L = \pi$), then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\},$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. && (16)\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0, 1, 2, \ldots. && (17)\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1, 2, 3, \ldots. && (18)
\end{aligned}$$

**Example 46.** *Let $f(x) = x^2$, $x \in [-\pi, \pi)$, and $f(x)$ be $2\pi$-periodic. Compute the Fourier coefficients.*

Since $f$ is *even* ($f(x) = f(-x)$ for all $x$), then $b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2,$$

and for $n \geq 1$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx\\
&= \frac{1}{n\pi}\left\{ x^2\sin(nx)\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin(nx)\,dx \right\}\\
&= \frac{(-1)^n\cdot 4}{n^2}.
\end{aligned}$$

Thus for $x \in (-\pi, \pi)$,

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(nx).$$

**Example 47.** *Let*

$$f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in (0, \pi). \end{cases},$$

*and let $f(x)$ be $2\pi$-periodic. Find the Fourier series of $f(x)$.*

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} 1dx\right) = 1,\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_{0}^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^{\pi} = 0,\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_{0}^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^{\pi}\\
&= \frac{1}{n\pi}(1 - (-1)^n) = \begin{cases} \frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}
\end{aligned}$$

Hence the Fourier series for $f(x)$ is:

$$\begin{aligned} f(x) &\sim \frac{1}{2} + \sum_{odd\ n} \frac{2}{n\pi}\sin(nx) = \\ &= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi}\sin(2n+1)x, \quad \forall x \in (-\pi, \pi). \end{aligned}$$

**General $2L$-periodic functions:**

**Example 48.** *Let $f(x)$ be 2-periodic and* $f(x) = \begin{cases} x, & 0 < x < 1; \\ 0, & -1 < x \le 0. \\ 0.5, & x = -1, 1. \end{cases}$

In this case, $L = 1$.

$$\begin{aligned} a_0 &= \int_0^1 x\,dx = \frac{1}{2}, \\ a_n &= \int_0^1 x\cos(n\pi x)\,dx = \frac{(-1)^n - 1}{(n\pi)^2}, \\ b_n &= \int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^n}{n\pi}. \end{aligned}$$

The full Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^n}{n\pi}\sin(n\pi x)\right].$$

**Example 49.** *Let $f(x)$ be 2-periodic and* $f(x) = \left\{ \begin{array}{ll} 3, & 0 \le x < 1; \\ \sin(\pi x), & -1 \le x < 0. \end{array} \right\}$ *Find $a_2$.*

**Points of discontinuity and convergence**

In equation (1), "=" means that the series on the right converges to the function on the left at each point $x$. It often happens that the Fourier series of a function f fails to converge to that function, in particular at the points of discontinuity of f.

The facts are:

- **The Fourier Convergence Theorem**: If the function $f(x)$ is piecewise continuously differentiable then its Fourier series converges for every x to the average value

$$f_{av}(x) = \frac{f(x+) + f(x-)}{2}, \tag{19}$$

  where

$$f(x+) = \lim_{t \to x+} f(t), \quad f(x-) = \lim_{t \to x-} f(t).$$

- At the points where $f(x)$ is continuous, $f_{av}(x) = f(x)$.

All the functions we shall consider in the sequel are piecewise continuously differentiable, and therefore the Fourier series will represent the function. In order to ensure that the Fourier series of function $f(x)$ converges to that function at every $x \in \mathbb{R}$, sometimes it is necessary to redefine $f(x)$ at the points of discontinuity x, so that $f_{av}(x) = f(x)$.

**Example 50.** *Let*

$$f(x) = \begin{cases} -1, & \text{for } x \in (-\pi, 0); \\ x, & \text{for } x \in (0, \pi); \\ 0, & \text{for } x = 0, \pi, -\pi. \end{cases}$$

*Determine the sums to which the series converges at* $x = 0, \pm\pi, 88\pi, 101\pi$.

**Solution:** The sum$=-0.5$ at 0; $\frac{\pi-1}{2}$ at $\pm\pi$.

**Geometric interpretation of Fourier series**

In the example above, if we let

$$S_1 = \frac{1}{2} + \frac{2}{\pi} \sin x,$$

$$S_3 = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x,$$

$$S_5 = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x,$$

each partial sum is a continuous function that approximates the discontinuous function f(x) on the interval $(-\pi, \pi)$. The bigger n, the better the approximation.

## 3.2 Functions Defined on Finite Intervals

### 1. Half-range Expansions

Let $f(x)$ be define on $(0, L)$. Three special extensions are important:

(i) Extend $f(x)$ as an odd function on $(-L, L)$ with period $2L$:

$$\tilde{f}(x) = f_{odd}(x) = \begin{cases} f(x), & x \in (0, L); \\ -f(-x), & x \in (-L, 0). \end{cases}$$

Then $a_n = 0$ for all $n$. Thus

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{20}$$

which is called **Fourier sine series** of $f$, where

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots.$$

(ii) Extend $f(x)$ as an even function on $(-L, L)$ with period $2L$:

$$\tilde{f}(x) = f_{even}(x) = \begin{cases} f(x), & x \in (0, L); \\ f(-x), & x \in (-L, 0). \end{cases}$$

Then $b_n = 0$ for all $n$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{21}$$

which is called **Fourier cosine series** of $f$, where

$$a_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})\,dx \qquad m = 0, 1, 2, 3, \ldots.$$

**The cosine and sine series here are known as HALF-RANGE EXPANSIONs.**

**Example 51.** *Let $f(x) = 3 + 2x$, $0 \le x \le 2$. Find the Fourier sine series, Fourier cosine series and their values at $x = 2$.*

**Example 52.** *Let* $f(x) = \begin{cases} x, & 0 \le x < 1. \\ 0, & 1 \le x < 2; \end{cases}$.

*(a) Find the Fourier sine series and Fourier cosine series.*

*(b) Find the odd and even extensions.*

*(c) Find the values of the sine and cosine series at $x = 2, 1, -1$.*

Solution: (a) (i) Fourier sine series: for $m = 1, 2, 3, \ldots,$

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\sin(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\sin(\frac{m\pi x}{2})\,dx = \left[-\frac{2}{m\pi}x\cos(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi x}{2})\right]_0^1 \\ &= -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

The Fourier sine series is

$$f(x) = \sum_{m=1}^{\infty} \left[ -\frac{2}{m\pi} \cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2} \sin(\frac{m\pi}{2}) \right] \sin(\frac{m\pi x}{2}).$$

(ii) Fourier cosine series: for $m = 0, 1, 2, 3, \dots,$

$$\begin{aligned} a_m &= \frac{2}{L} \int_0^L f(x) \cos(\frac{m\pi x}{L})\, dx = \frac{2}{2} \int_0^2 f(x) \cos(\frac{m\pi x}{2})\, dx \\ &= \int_0^1 x \cos(\frac{m\pi x}{2})\, dx = \left[ \frac{2}{m\pi} x \sin(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2} \cos(\frac{m\pi x}{2}) \right]_0^1 \\ &= \frac{2}{m\pi} \sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2} \cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}. \end{aligned}$$

**2. The Fourier Series on Interval** $[a, b]$

Let $f(x)$ be defined on $[a, b]$. Then we can extend $f(x)$ to be a periodic function $\tilde{f}(x)$ with period $b - a$. Let $2L = b - a$, then $L = \frac{b-a}{2}$.

**Definition 8.** *The Fourier Series of $f(x)$ on $(a, b)$ is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})) \tag{22}$$

*at every $x \in [a, b]$ where $f$ is continuous. The coefficients $a_n$ $(n \geq 0)$ and the coefficients $b_n$ $(n \geq 1)$ are calculated as follows:*

$$a_n = \frac{1}{L} \int_a^b f(x) \cos(\frac{n\pi x}{L})\, dx, \;\; n = 0, 1, 2, \dots; \quad b_n = \frac{1}{L} \int_a^b f(x) \sin(\frac{n\pi x}{L})\, dx, \;\; n = 1, 2, 3, \dots.$$

**Example 53.** *Let* $f(x) = \begin{cases} x, & 0 \leq x < 1. \\ 0, & 1 \leq x < 2; \end{cases}$.

*(a) Find the Fourier series.*

*(b) Find the values of the series at $x = 2, 1, -1$.*

**Solution:** 2-periodic extension of $f$:

$$\tilde{f}(x) = \begin{cases} x, & 0 \leq x < 1; \\ 0, & 1 \leq x < 2. \\ \tilde{f}(x+2) = \tilde{f}(x), & \text{for any } x. \end{cases}$$

In this case, $L = 1$.

$$\begin{aligned} a_0 &= \int_0^1 x\,dx = \frac{1}{2}, \\ a_m &= \int_0^1 x\cos(m\pi x)\,dx = \frac{(-1)^m - 1}{(m\pi)^2}, \\ b_m &= \int_0^1 x\sin(m\pi x)\,dx = \frac{(-1)^m}{m\pi}. \end{aligned}$$

The full Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^n}{n\pi}\sin(n\pi x)\right].$$

At $x = 2$, this series converges to

$$\frac{\tilde{f}(2+) + \tilde{f}(2-)}{2} = \tilde{f}(2) = 0;$$

At $x = 1$, this series converges to

$$\frac{\tilde{f}(1+) + \tilde{f}(1-)}{2} = \frac{1+0}{2} = 0.5$$

At $x = -1$, this series converges to

$$\frac{\tilde{f}(-1+) + \tilde{f}(-1-)}{2} = \frac{1+0}{2} = 0.5.$$

**Example 54.** *Let* $f(x) = \begin{cases} 3, & -1 \le x < 0; \\ 2x, & 0 \le x < 3; \end{cases}$ *Find its Fourier series.*

**Solution:** $L = 2$.

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{3} f(x)dx = 6, \\ a_n &= \frac{1}{2}\int_{-1}^{3} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{n\pi}\left(-3\sin\frac{n\pi}{2} + \frac{4}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n\pi}\right), \\ b_n &= \frac{1}{2}\int_{-1}^{3} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{n\pi}\left(-3 - 3\cos\frac{n\pi}{2} - \frac{4}{n\pi}\sin\frac{n\pi}{2}\right). \end{aligned}$$

# 4 Partial Differential Equations

A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. For example, Laplace's equation for $\Phi(x, y)$,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{23}$$

arises in many places in mathematics and physics. For simplicity, we will use subscript notation for partial derivatives, so this equation can also be written $\Phi_{xx} + \Phi_{yy} = 0$.

We say a function is a **solution** to a PDE if it satisfy the equation and any side conditions given. Mathematicians are often interested in if a solution **exists** and when it is **unique**.

Example. $\Phi_1 = x$ and $\Phi_2 = x^2 - y^2$ are solutions to Laplace's equation (23).

The **order** of PDE is the highest partial derivative that appears in the equation. So, Laplace's Equation (23) is second-order.

We also define **linear** PDE's as equations for which the dependent variable (and its derivatives) appear in terms with degree at most one. Anything else is called **nonlinear**.

Linear equations can further be classified as **homogeneous** for which the dependent variable (and it derivatives) appear in terms with degree **exactly** one, and non-homogeneous which may contain terms which only depend on the independent variable.

If two solutions, say $u_1$ and $u_2$, satisfy a linear homogeneous PDE, that any linear combination of them

$$u = c_1 u_1 + c_2 u_2$$

is also a solution. So, for example, since

$$\Phi_1 = x^2 - y^2 \qquad \Phi_2 = x$$

both satisfy Laplace's equation, $\Phi_{xx} + \Phi_{yy} = 0$, so does any linear combination of them

$$\Phi = c_1\Phi_1 + c_2\Phi_2 = c_1(x^2 - y^2) + c_2 x.$$

This property is extremely useful for constructing solutions which satisfy certain initial conditions and boundary conditions.

## 4.1 The Heat Equation

When we derived Newton's Law of cooling we made several assumptions – most importantly that the temperature does not vary with location. If we account for the variation of temperature with location, we can derive a PDE called the **heat equation** or, more generally, the **diffusion equation**. Suppose we consider a metal bar, with a uniform cross-sectional area, $A$, whose temperature, $u(x,t)$, is a function of time, $t$, and the position, $x$, along the bar (that is we assume the temperature is uniform in every cross-section). Then $u(x,t)$ satisfies the diffusion equation (DE),

$$u_t = c^2 u_{xx}, \tag{24}$$

where $c^2$ is a constant known as the **thermal diffusivity**, determined by the geometry and physical properties of the metal bar.

Suppose we consider a bar of finite length $L$, occupying the region $0 < x < L$. At the boundaries of the metal bar we can specify a fixed temperature,

$$u(0,t) = u_0 \qquad u(L,t) = u_1, \tag{25}$$

which are usually referred to as Dirichlet **boundary conditions (BCs)**.

We also need to specify the initial temperature distribution,

$$\textbf{Initial condition (IC)}: \quad u(x,0) = f(x) \qquad 0 \le x \le L. \tag{26}$$

#### 4.1.1 The Bar with Zero Boundary Conditions

**Theorem 13.** *The solution of the heat equation*

$$\begin{aligned} u_t &= c^2 u_{xx} && 0 < x < L, t > 0 && PDE \\ u(0,t) = 0 \quad u(L,t) &= 0 && t > 0 && BC \\ u(x,0) &= f(x) && 0 < x < L, && IC \end{aligned}$$

*is given by*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t},$$

*where*

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{27}$$

*Proof.* The method to solve the wave equation comprises three steps.

**Step 1: Separation of variables.**

Let $u(x,t) = X(x)T(t)$ and substitute it into the equation we have

$$XT_t = c^2 X_{xx} T$$

and dividing by $XT$ we find

$$\frac{T_t}{c^2T} = \frac{X_{xx}}{X} = \lambda. \tag{28}$$

where $\lambda$ is to be determined. Now because $T_t/DT$ is **only** a function of $t$ and $X_{xx}/X$ is **only** a function of $x$ we know that $\lambda$ must be independent of $x$ and $t$ respectively, and therefore must be a constant – consequently it is known as the **separation constant**. We can now solve the resulting ODE for $T(t)$

$$T_t = \lambda c^2 T \quad \Rightarrow \quad T(t) = Ce^{\lambda Dt}. \tag{29}$$

**Step 2: Eigenfunction.**

We now look for a solution for the $X(x)$ equation that also satisfies the homogeneous boundary conditions. From the boundary conditions (BC), we know that

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 \tag{30}$$

$$u(L,t) = X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0 \tag{31}$$

So finally we conclude that we are looking for solutions to the **Boundary Value Problem** for $X(x)$,

$$X_{xx} - \lambda X = 0, \qquad X(0) = X(L) = 0. \tag{32}$$

Now we consider three cases:

1. If $\lambda = 0$, then $X(x) = Ax + B$, it follows from $X(0) = 0$ that $B = 0$. Thus $X(x) = Ax$. From $X(L) = 0$ we have $A = 0$, so $X(x) = 0$, trivial solution.

2. If $\lambda > 0$, then $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$, it follows from $X(0) = X(L) = 0$ that $A + B = 0$, $Ae^{\sqrt{\lambda}L} + Be^{-\sqrt{\lambda}L} = 0$. Thus $A = B = 0$ and so $X(x) \equiv 0$, a trivial solution.

3. If $\lambda < 0$, then $X(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$. Applying the boundary conditions we see that $X(0) = 0$ implies that $A = 0$, and that $B\sin(\sqrt{-\lambda}L) = 0$. To have non-trivial solution, $B \neq 0$, so $\sin(\sqrt{-\lambda}L) = 0, \Rightarrow \sqrt{-\lambda} = \frac{n\pi}{L} \Rightarrow$

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, ...$$

   where $B_n$ are constants. Note that $\lambda = -(\frac{n\pi}{L})^2$. These special values of $\lambda$ are called **eigenvalues** and the associated functions, $X_n(x)$, are known as **eigenfunctions**. This implies

$$T_n(t) = C_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}.$$

Multiplying the solution for $X_n(x)$ and $T_n(t)$ together finally yields a solution for $u_n(x,t)$ satisfying BCs $u_n(0,t) = u_n(L,t) = 0$ is

$$u_n(x,t) \equiv b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \quad n = 1, 2, 3 \ldots. \tag{33}$$

**Step 3: Superposition.** We use superposition of the eigenfunctions to satisfy the initial condition.

To consider the IC: $u(x,0) = f(x)$, we let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}, \Rightarrow$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \Rightarrow$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

♠

**Example 55.** *Solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

*subject to the boundary conditions*

$$u(0,t) = u(1,t) = 0,$$

*and the initial condition*

$$u(x,0) = f(x) = \begin{cases} x & \text{if } 0 < x < 1/2, \\ 1-x & \text{if } 1/2 < x < 1. \end{cases}$$

Solution: $c = 1$, $L = 1$. Thus

$$b_n = 2\left(\int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 (1-x)\sin(n\pi x) dx\right). \tag{34}$$

Integration by parts yields,

$$b_{2m} = (-1)^m \frac{1}{m^2\pi^2}, \quad b_{2m+1} = 0,$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} = \sum_{m=1}^{\infty} (-1)^m \frac{1}{m^2\pi^2} \sin(2m\pi x) e^{-(2m\pi)^2 t}$$

### 4.1.2 The Bar with Non-zero Boundary Conditions

THE DIRICHLET PROBLEM FOR THE DIFFUSION EQUATION
(NON-HOMOGENEOUS BOUNDARY CONDITIONS)

$$\begin{array}{rrcl} & u_t = \alpha^2 u_{xx} & 0 < x < L, t > 0 & PDE \\ u(0,t) = A & u(L,t) = B & t > 0 & BC \\ & u(x,0) = f(x) & 0 \le x \le L. & IC \end{array}$$

By letting

$$u(x,t) = v(x) + w(x,t),$$

we see that $w(x,t)$ satisfies Homogeneous Boundary Conditions and $v(x)$ satisfies

$$v''(x) = 0, \quad v(0) = A, v(L) = B.$$

**Theorem 14.** *The solution of the heat equation (24) subject to BCs: $u(0,t) = A$, $u(L,t) = B$ and IC: $u(x,0) = f(x)$ is given by*

$$u(x,t) = v(x) + w(x,t)$$

*with*

$$v(x) = \frac{B-A}{L}x + A, \quad w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t},$$

*where $b_n$ satisfies*

$$b_n = \frac{2}{L}\int_0^L [f(x) - v(x)] \sin\left(\frac{n\pi x}{L}\right) dx. \tag{35}$$

Example. Find the solution of the Heat Equation $u_{xx} = 9u_t$, $0 < x < 2$, $t > 0$, subject to BCs: $u(0,t) = 1$, $u(2,t) = 3$ and IC: $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 2x+1, & 0 \le x < 1, \\ x+1, & 1 \le x < 2, \end{cases}$$

Solution: Here $\alpha = \frac{1}{3}$, $L = 2$, $A = 1$, $B = 3$. Thus $v(x) = x + 1$. By (35), for $m = 1, 2, 3, \ldots$,

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L [f(x) - v(x)] \sin(\frac{m\pi x}{L})\, dx = \frac{2}{2}\int_0^2 [f(x) - v(x)] \sin(\frac{m\pi x}{2})\, dx \\ &= \int_0^1 x \sin(\frac{m\pi x}{2})\, dx = \left[-\frac{2}{m\pi} x \cos(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2} \sin(\frac{m\pi x}{2})\right]_0^1 \\ &= -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

$$u(x,t) = x + 1 + \sum_{n=1}^{\infty} \left[ -\frac{2}{n\pi} \cos(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2} \sin(\frac{n\pi}{2}) \right] \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{6}\right)^2 t}.$$

### 4.1.3 The Bar With Insulated Ends

$$\begin{aligned} u_t &= \alpha^2 u_{xx} & 0 < x < L, t > 0 & \quad PDE \\ u_x(0,t) = u_x(L,t) &= 0 & t > 0 & \quad BC \\ u(x,0) &= f(x) & 0 \le x \le L. & \quad IC \end{aligned}$$

The solution of this PDE is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \tag{36}$$

where $a_n$ satisfies

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx. \tag{37}$$

Example. Find the solution of the Heat Equation $u_{xx} = u_t$, $0 < x < 2$, $t > 0$, subject to BCs: $u_x(0,t) = u_x(2,t) = 0$ and IC: $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 2, & 0 \le x < 1, \\ 1, & 1 \le x < 2, \end{cases}.$$

Example. Find the solution of the Heat Equation $u_{xx} = u_t$, $0 < x < 2$, $t > 0$, subject to $u_x(0,t) = u_x(2,t) = 0$ and

$$u(x,0) = 2\cos(2\pi x) - 3\cos(5\pi x).$$

## 4.2 The Wave Equation

Consider a string stretched along the $x$-axis with its ends held fixed at $x = 0$ and $x = L > 0$. If the string is plucked, then it will vibrate. Denote by $u(x,t)$ the vertical displacement of the string at location $x$ and time $t$. Then $u(x,t)$ is governed by the wave equation with the form:

$$u_{tt} = c^2 u_{xx}, \tag{38}$$

where $u = u(x,t)$ can be thought of as the vertical displacement of the vibration of a string.

If we tie the string at both ends we can have the following boundary conditions:

$$u(0,t) = A(t), u(L,t) = B(t),$$

where $A, B$ are piecewise functions. For example, we can have a sinusoidal function at one end and a Heaviside function at the other.

When the boundary values $A$ and $B$ are 0 we obtain the **Dirichlet Problem** for the wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < L, t > 0 & \quad PDE \\ u(0,t) = 0, u(L,t) &= 0, & t > 0 & \quad BC \\ u(x,0) = f(x), u_t(x,0) &= g(x) & 0 < x < L & \quad IC. \end{aligned}$$

As you have seen in Heat equation, the method of separating the variables is a very convenient way to obtain solutions for PDEs. In the case of the Dirichlet Problem we will quickly review the method.

**Theorem 15.** *The solution of the above Dirichlet Problem (wave problem) is given by:*

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

*where:*

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Proof. The method to solve the wave equation comprises three steps.

**Step 1: Separation of variables.**

Let $u(x,t) = X(x)T(t)$ and substitute it into the equation $u_{tt} = c^2 u_{xx}$, to obtain:

$$X(x)T''(t) = c^2 X''(x)T(t), \Rightarrow$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = constant.$$

(the equality is one of functions of different variables, so both quotients have to be constant.)

We have the following three cases: $constant = -\lambda^2$, $\lambda^2$, 0.

**Step 2: Eigenfunctions.**

**Case 1** When the constant is $\lambda^2$, then the solutions for $\frac{X''(x)}{X(x)} = \lambda^2$, are: $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$, and the solutions for $\frac{T''(t)}{c^2 T(t)} = \lambda^2$, are $T(t) = d_1 e^{\lambda ct} + d_2 e^{-\lambda ct}$. Then

$$u(x,t) = (d_1 e^{\lambda ct} + d_2 e^{-\lambda ct})(c_1 e^{\lambda x} + c_2 e^{-\lambda x}).$$

Let's take a look at the boundary conditions: $u(0,t) = 0, u(L,t) = 0$. We imply that $c_1 = c_2 = 0$, and so $u(x,t) = 0$, only trivial solution.

**Case 2** When the constant is 0, then the equations become $X''(x) = T''(t) = 0$, and $X(x) = c_1 x + c_2$, and $T(t) = d_1 t + d_2$. Then

$$u(x,t) = (d_1 t + d_2)(c_1 x + c_2).$$

Let's take a look at the boundary conditions: $u(0,t) = 0, u(L,t) = 0$. We imply that $c_1 = c_2 = 0$, and so $u(x,t) = 0$, only trivial solution.

**Case 3** When the constant is $-\lambda^2$, then the solutions for $\frac{X''(x)}{X(x)} = -\lambda^2$, are:

$$X(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x),$$

and the solutions for $\frac{T''(t)}{c^2 T(t)} = -\lambda^2$, are:

$$T(t) = d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct).$$

Then

$$u(x,t) = (d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct))(c_1 \sin(\lambda x) + c_2 \cos(\lambda x)).$$

The boundary conditions $u(0,t) = u(L,t) = 0$ translate into:

$$\begin{aligned} (d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct))(c_1 \sin(0) + c_2 \cos(0)) &= 0 \\ (d_1 \sin(\lambda ct) + d_2 \cos(\lambda ct))(c_1 \sin(\lambda L) + c_2 \cos(\lambda L)) &= 0, \quad \forall t > 0, \end{aligned}$$

namely:

$$\begin{aligned} c_2 &= 0 \\ c_1 \sin(\lambda L) &= 0. \end{aligned}$$

From the last condition we obtain $\lambda = \frac{\pi n}{L}$, and

$$u_n(x,t) = \left[ d_{1n} \sin(\frac{\pi n}{L} ct) + d_{2n} \cos(\frac{\pi n}{L} ct) \right] c_n \sin(\frac{\pi n}{L} x).$$

**Step 3: Superposition.** We use superposition of the eigenfunctions to satisfy the initial condition.

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) &= \sum_{n=1}^{\infty} \left[ d_{1n} \sin(\frac{\pi nct}{L}) + d_{2n} \cos(\frac{\pi nct}{L}) \right] c_n \sin(\frac{\pi nx}{L}) \\ &= \sum_{n=1}^{\infty} \left[ a_n \cos(\frac{\pi nct}{L}) + b_n \sin(\frac{\pi nct}{L}) \right] \sin(\frac{\pi nx}{L}). \end{aligned}$$

The only conditions left to check are the initial conditions:

$$\begin{aligned} u(x,0) &= f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \\ u_t(x,0) &= g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

From the 1st one we have

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

From the 2nd one we have

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Then

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})\right]\sin(\frac{n\pi x}{L}).$$

**Example 56.** *Solve the equation* $9u_{tt} = u_{xx},\ 0 < x < 4, t > 0$ *subject to* $u(0,t) = 0, u(4,t) = 0$ *for* $t > 0$*, and*

$$u(x,0) = \begin{cases} 2x, & 0 \le x \le 2 \\ -2x+4, & 2 < x \le 4 \end{cases}$$

$u_t(x,0) = 0\ (0 < x < 4)$.

Solution: $c = 1/3$, $L = 4$. By $u_t(x,0) = 0$ we obtain $b_n = 0$.

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx = \int_0^2 x\sin(\frac{n\pi x}{4})dx + \int_2^4 (-x+2)\sin(\frac{n\pi x}{4})dx \\ &= \left[-\frac{4}{n\pi}x\cos(\frac{n\pi x}{4}) + \frac{16}{n^2\pi^2}x\sin(\frac{n\pi x}{4})\right]_0^2 \\ &\quad + \left[-\frac{4}{n\pi}(-x+2)\cos(\frac{n\pi x}{4}) - \frac{16}{n^2\pi^2}x\sin(\frac{n\pi x}{4})\right]_2^4 \\ &= -\frac{8}{n\pi}\cos\frac{n\pi}{2} + \frac{32}{\pi^2 n^2}\sin(\frac{n\pi}{2}) + \frac{(-1)^n 8}{n\pi}. \end{aligned}$$

Now we can write the formal solution to the plucked string equation:

$$u(x,t) = \sum_{n=1}^{\infty}\left[-\frac{8}{n\pi}\cos\frac{n\pi}{2} + \frac{32}{\pi^2n^2}\sin(\frac{n\pi}{2}) + \frac{(-1)^n8}{n\pi}\right]\cos(\frac{n\pi t}{12})\sin(\frac{n\pi x}{4}).$$

**Example 57.** *Solve the following equation:*

$$\begin{aligned} u_{tt} &= 9u_{xx}, & 0 < x < 2, t > 0 & \quad DE \\ u(0,t) = 0, u_x(2,t) &= 0, & t > 0 & \quad BC \\ u(x,0) = 2\sin(3\pi x), u_t(x,0) &= \sin(5\pi x) & 0 < x < 2 & \quad IC. \end{aligned}$$

Solution: $c = 3, L = 2$.

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos(\frac{3n\pi t}{2}) + b_n \sin(\frac{3n\pi t}{2}) \right] \sin(\frac{n\pi x}{2}).$$

$$u(x,0) = 2\sin(3\pi x) \Rightarrow \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{2}) = 2\sin(3\pi x) \Rightarrow$$

$$a_6 = 2, \quad a_n = 0 (n \neq 6).$$

$$u_t(x,0) = \sin(5\pi x) \Rightarrow \sum_{n=1}^{\infty} b_n \frac{3n\pi}{2} \sin(\frac{n\pi x}{2}) = \sin(5\pi x), \Rightarrow$$

$$b_{10} = \frac{1}{15\pi}, \quad b_n = 0 (n \neq 10).$$

Thus

$$u(x,t) = 2\cos(9\pi t)\sin(3\pi x) + \frac{1}{15\pi} \sin(15\pi t)\sin(5\pi x).$$

## 4.3 Laplace's Equation

Now we consider the following **Dirichlet Problem** for the Laplace equation:

$$\begin{aligned} PDE: &\quad \nabla^2 u = \tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2} = \quad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\quad u(x,y) = f(x,y), \quad (x,y) \in \partial R. \end{aligned}$$

R is a region. We can consider $u(x,y)$ as the steady-state (time-independent) temperature distribution.

### 4.3.1 Solutions within Rectangular Regions

Let

$$R = \{(x,y) : 0 < x < L, 0 < y < M\}.$$

**Case 1**: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\quad u(x,0) = u(x,M) = u(0,y) = 0, u(L,y) = f(y). \end{aligned}$$

We will use the method of separation of variables again.

Let $u(x,y) = X(x)Y(y)$ and substitute in the equation to obtain:

$$\frac{X''}{X} = -\frac{Y''}{Y} = constant, say, k.$$

We have the following three cases: $k = -\lambda^2$, $\lambda^2$, 0, where $\lambda > 0$.

**(i)** $k = 0$. Then the equations become $X''(x) = T''(t) = 0$, and $X(x) = c_1 x + c_2$, and $T(t) = d_1 t + d_2$. Then

$$u(x,t) = (d_1 t + d_2)(c_1 x + c_2).$$

By the boundary conditions: $u(x,0) = 0, u(x,M) = 0$, we imply that $d_1 = d_2 = 0$, and so $u(x,t) = 0$, only trivial solution.

**(ii)** $k = -\lambda^2$. Then the solutions for $\frac{X''(x)}{X(x)} = -\lambda^2$, are: $X(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$, and the solutions for $-\frac{Y''}{Y} = -\lambda^2$, are $Y(y) = d_1 e^{\lambda y} + d_2 e^{-\lambda y}$. Then

$$u(x,t) = [c_1\cos(\lambda x) + c_2\sin(\lambda x)][d_1 e^{\lambda y} + d_2 e^{-\lambda y}].$$

By the boundary conditions: $u(x,0) = 0, u(x,M) = 0$, we imply that $d_1 = d_2 = 0$, and so $u(x,t) = 0$, only trivial solution.

**(iii)** $k = \lambda^2$. Then

$$\Rightarrow \begin{cases} X'' - \lambda^2 X = 0 \\ Y'' + \lambda^2 Y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \\ Y = c_3\sin\lambda y + c_4\cos\lambda y \end{cases}$$

$$\Rightarrow u(x,y) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3\sin\lambda y + c_4\cos\lambda y).$$

The boundary conditions

$$u(x,0) = 0 \Rightarrow c_4 = 0; \quad u(x,M) = 0 \Rightarrow c_3\sin\lambda M = 0;$$

$$u(0,y) = 0, \Rightarrow c_2 = -c_1.$$

$$\Rightarrow u(x,y) = c_1(e^{\lambda x} - e^{-\lambda x})c_3\sin\lambda y = 2c_1c_3\sinh(\lambda x)\sin\lambda y.$$

To have non-trivial solution, $\sin\lambda M = 0, \Rightarrow \lambda = \frac{n\pi}{M}$, where $n$ is integer.

$$\Rightarrow u_n(x,y) = a_n\sinh\left(\frac{n\pi x}{M}\right)\sin\left(\frac{n\pi y}{M}\right) \Rightarrow .$$

$$u(x,y) = \sum_{n=1}^{\infty} a_n\sinh\left(\frac{n\pi x}{M}\right)\sin\left(\frac{n\pi y}{M}\right). \tag{39}$$

By the boundary condition

$$f(y) = u(L,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi L}{M}\right) \sin\left(\frac{n\pi y}{M}\right)$$

we imply that

$$a_n = \frac{2}{M \sinh\left(\frac{n\pi L}{M}\right)} \int_0^M f(y) \sin\left(\frac{n\pi y}{M}\right) dy. \tag{40}$$

**Case 2**: Consider the boundary-value problem

$$\begin{aligned} PDE: &\qquad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\qquad u(x,0) = u(x,M) = u(L,y) = 0, u(0,y) = f(y). \end{aligned}$$

Let $z = L - x, \quad u(x,y) = w(z,y)$. Then $x = 0 \leftrightarrow z = L; x = L \leftrightarrow z = 0$ and the equation becomes:

$$\begin{aligned} PDE: &\qquad w_{zz} + w_{yy} = 0, \quad (z,y) \in R; \\ BC: &\qquad w(z,0) = w(z,M) = w(0,y) = 0, w(L,y) = f(y). \end{aligned}$$

According to Case 1, the solution is:

$$w(z,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi z}{M}\right) \sin\left(\frac{n\pi y}{M}\right)$$

with

$$a_n = \frac{2}{M \sinh\left(\frac{n\pi L}{M}\right)} \int_0^M f(y) \sin\left(\frac{n\pi y}{M}\right) dy, \quad n \geq 1. \tag{41}$$

Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi(L-x)}{M}\right) \sin\left(\frac{n\pi y}{M}\right). \tag{42}$$

**Case 3**: Consider the boundary-value problem

$$\begin{aligned} PDE: &\qquad u_{xx} + u_{yy} = 0, \quad (x,y) \in R; \\ BC: &\qquad u(x,0) = u(0,y) = u(L,y) = 0, u(x,M) = f(x). \end{aligned}$$

Interchange $x$ and $y$ we will have Case 1. By Case 1, interchange $L$ and $M$,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right), \tag{43}$$

where

$$a_n = \frac{2}{L \sinh\left(\frac{n\pi M}{L}\right)} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{44}$$

**Case 4**: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx}+u_{yy}=0, \quad (x,y)\in R; \\ BC: &\quad u(x,M)=u(0,y)=u(L,y)=0, u(x,0)=f(x). \end{aligned}$$

This is similar to Case 2. Let $z=M-y, \quad u(x,y)=w(x,z)$. Then $y=0 \leftrightarrow z=M; y=M \leftrightarrow z=0$ and the equation becomes:

$$\begin{aligned} PDE: &\quad w_{xx}+w_{zz}=0, \quad (x,z)\in R; \\ BC: &\quad w(x,0)=w(0,y)=w(L,y)=0, w(L,M)=f(x). \end{aligned}$$

We have

$$w(x,z)=\sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n=\frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{45}$$

Thus

$$u(x,y)=\sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi(M-y)}{L}\right)\sin\left(\frac{n\pi x}{L}\right). \tag{46}$$

**Case 5**: Consider the boundary-value problem

$$\begin{aligned} PDE: &\quad u_{xx}+u_{yy}=0, \quad (x,y)\in R; \\ BC: &\quad u(x,0)=f_1(x), u(x,M)=f_2(x), u(0,y)=f_3(y), u(L,y)=f_4(y). \end{aligned}$$

Then we just need to combine cases 1–4 together.

**Theorem 16.** *The solution of the above Laplace equation is:*

$$u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y),$$

*where*

$$\begin{aligned}
u_1(x,y) &= \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi(M-y)}{L}\sin\left(\frac{n\pi x}{L}\right), a_n=\frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx, \\
u_2(x,y) &= \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right), b_n=\frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx, \\
u_3(x,y) &= \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi(L-x)}{M}\sin\left(\frac{n\pi y}{M}\right), c_n=\frac{2}{M\sinh\left(\frac{n\pi L}{M}\right)}\int_0^M f_3(y)\sin\left(\frac{n\pi y}{M}\right)dy, \\
u_4(x,y) &= \sum_{n=1}^{\infty} d_n \sinh\left(\frac{n\pi x}{M}\right)\sin\left(\frac{n\pi y}{M}\right), d_n=\frac{2}{M\sinh\left(\frac{n\pi L}{M}\right)}\int_0^M f_4(y)\sin\left(\frac{n\pi y}{M}\right)dy.
\end{aligned}$$

**Example** Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within

$$R = \{(x, y) : 0 < x < 3, 0 < y < 2\}$$

with

$$BC: \quad u(x, 0) = 2x + 2, u(x, 2) = 0, u(0, y) = 2 - y, u(3, y) = 8 - 4y.$$

Solution: $L = 3$, $M = 2$.

$$\begin{aligned}
d_n &= \frac{2}{2\sinh\left(\frac{3n\pi}{2}\right)} \int_0^2 (8 - 4y) \sin\left(\frac{n\pi y}{2}\right) dy = \frac{16}{n\pi \sinh\left(\frac{3n\pi}{2}\right)}, \\
c_n &= \frac{2}{2\sinh\left(\frac{3n\pi}{2}\right)} \int_0^2 (2 - y) \sin\left(\frac{n\pi y}{2}\right) dy = \frac{4}{n\pi \sinh\left(\frac{3n\pi}{2}\right)}, \\
a_n &= \frac{2}{3\sinh\left(\frac{2n\pi}{3}\right)} \int_0^3 (2x + 2) \sin\left(\frac{n\pi x}{3}\right) dx = \frac{-16\cos(n\pi) + 4}{n\pi \sinh\left(\frac{2n\pi}{3}\right)}, \\
b_n &= 0.
\end{aligned}$$

$$\begin{aligned}
u_4(x, y) &= \sum_{n=1}^{\infty} \frac{16}{n\pi \sinh\left(\frac{3n\pi}{2}\right)} \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right), \\
u_3(x, y) &= \sum_{n=1}^{\infty} \frac{4}{n\pi \sinh\left(\frac{3n\pi}{2}\right)} \sinh\left(\frac{n\pi(3 - x)}{2}\right) \sin\left(\frac{n\pi y}{2}\right), \\
u_1(x, y) &= \sum_{n=1}^{\infty} \frac{-16\cos(n\pi) + 4}{n\pi \sinh\left(\frac{2n\pi}{3}\right)} \sinh\left(\frac{n\pi(2 - y)}{3}\right) \sin\left(\frac{n\pi x}{3}\right), \\
u_2(x, y) &= 0.
\end{aligned}$$

### 4.3.2 Polynomial Solutions

**Theorem 17.** *(Polynomial solution) Consider the boundary-value problem*

$$\begin{aligned}
PDE: &\quad u_{xx} + u_{yy} = 0, \quad (x, y) \in R = \{(x, y) : 0 < x < L, 0 < y < M\}; \\
BC: &\quad u(x, 0) = f_1(x), u(x, M) = f_2(x), u(0, y) = f_3(y), u(L, y) = f_4(y).
\end{aligned}$$

*If all $f_i$ are continuous and linear on the boundary, then the PDE has a polynomial solution*

$$u(x, y) = ax + by + cxy + d.$$

**Example.** Find a polynomial solution of Laplace's equation within

$$R = \{(x, y) : 0 < x < 3, 0 < y < 2\}$$

with

$$BC: \quad u(x,0) = 2x + 2, u(x,2) = 0, u(0,y) = 2 - y, u(3,y) = 8 - 4y.$$

Solution: Note that all $f_i$ are continuous and linear on the boundary. Let

$$u(x,y) = ax + by + cxy + d.$$

$$u(x,0) = 2x + 2 \Rightarrow a = d = 2.$$

$$u(0,y) = 2 - y = by + 2 \Rightarrow b = -1.$$

$$u(3,y) = 8 - 4y = 6 - y + 3cy + 2 \Rightarrow c = -1.$$

$u(x,y) = 2x - y - xy + 2.$

### 4.3.3 Regions with Circular Boundaries

**Solutions Inside a circle**

In this case, we use polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Then we have the boundary value problem:

$$\begin{aligned} PDE: &\qquad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < R; \\ BC: &\qquad u(R,\theta) = f(\theta). \end{aligned}$$

**Theorem 18.** *Let $f$ be continuous, $2\pi$-periodic, $f'$ be piecewise continuous. Then the solution of the PDE above is*

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

*where*

$$a_n = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n \geq 1.$$

Proof. Let $u(r,\theta) = F(r)G(\theta)$. We imply that

$$r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = k.$$

Thus we obtain the Euler-Cauchy equation

$$r^2 F''(r) + rF'(r) - kF(r) = 0$$

and the harmonic equation

$$G''(\theta) + kG(\theta) = 0.$$

Since $f(\theta)$ is $2\pi$-periodic, $G(\theta)$ should be $2\pi$-periodic. Thus $k > 0$ and $k = n^2$, $n = 1, 2, ....$ Then

$$G_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), F(r) = C_n r^n + D_n r^{-n}.$$

The eigenfunctions are

$$u_n(r, \theta) = \left(C_n r^n + D_n r^{-n}\right) (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

By superposition,

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(C_n r^n + D_n r^{-n}\right) (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

The bounded interior solution is ($D_n = 0$ since $u(0, \theta)$ is bounded):

$$u(r, \theta) = \sum_{n=0}^{\infty} C_n r^n (A_n \cos(n\theta) + B_n \sin(n\theta))$$

$$= \sum_{n=0}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

The boundary condition implies that

$$u(R, \theta) = \sum_{n=0}^{\infty} R^n (a_n \cos(n\theta) + b_n \sin(n\theta)) = f(\theta).$$

Thus

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta.$$

**Example.** Solve

$$\begin{aligned} PDE: &\quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad r < 3; \\ BC: &\quad u(3, \theta) = 27 \cos(2\theta) - 54 \sin(3\theta). \end{aligned}$$

Solution: $R = 3$. The boundary condition implies that

$$u(3, \theta) = \sum_{n=0}^{\infty} 3^n (a_n \cos(n\theta) + b_n \sin(n\theta)) = 27 \cos(2\theta) - 54 \sin(3\theta).$$

Thus $3^2 a_2 = 27$, $\quad 3^3 b_3 = -54$, other $a_n$ and $b_n$ are 0, so then

$$a_n = \begin{cases} 3, & \text{if } n = 2; \\ 0, & \text{if } n \neq 2. \end{cases}$$

$$b_n = \begin{cases} -2, & \text{if } n = 3; \\ 0, & \text{if } n \neq 3. \end{cases}$$

$$u(r, \theta) = 3r^2 \cos(2\theta) - 2r^3 \sin(3\theta).$$

**Solutions Outside a circle**

Consider the boundary value problem:

$$\begin{aligned} PDE: & \qquad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad r > a; \\ BC: & \qquad u(a, \theta) = f(\theta). \end{aligned}$$

**Theorem 19.** *Let $f$ be continuous, $2\pi$-periodic, $f'$ be piecewise continuous. Let $u(r, \theta)$ be bounded in $R$. Then the solution of the PDE above is*

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

*where*

$$a_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad n \geq 0,$$

$$b_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta, \quad n \geq 1.$$

**Example.** Find the bounded solution of $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ outside the circle $r = 2$, subject to the boundary condition $u(2, \theta) = 4\cos^2(2\theta) - 5\sin(3\theta)$.

Solution: $a = 2$. The solution of the PDE above is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Note that $4\cos^2(2\theta) = 2[\cos(4\theta) + 1]$. Thus

$$u(2, \theta) = 2 + 2\cos(4\theta) - 5\sin(3\theta).$$

Therefore

$$2 + 2\cos(4\theta) - 5\sin(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

which implies that

$a_0 = 4, 2 = 2^{-4}a_4, a_n = 0$ for $n \neq 0, 4$; $-5 = 2^{-3}b_3$, $b_n = 0$ for $n \neq 3$. Hence
$a_0 = 4, a_4 = 32, a_n = 0$ for $n \neq 0, 4$; $b_3 = -40$, $b_n = 0$ for $n \neq 3$. Hence

$$u(r, \theta) = 2 + 32r^{-4}\cos(4\theta) - 40r^{-3}\sin(3\theta).$$

**Solutions within an Annulus**

Consider the boundary value problem:

$$\begin{aligned} PDE: &\quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a < r < b; \\ BC: &\quad u(a, \theta) = f(\theta), u(b, \theta) = g(\theta). \end{aligned}$$

This is a combination of the above two sections.

**Theorem 20.** *Let $f$ and $g$ be continuous, $2\pi$-periodic, $f'$ and $g'$ be piecewise continuous. Then the solution of the PDE above is*

$$u(r, \theta) = \frac{a_0 + b_0 \ln r}{2} + \sum_{n=1}^{\infty}[(a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)],$$

*where*

$$\begin{aligned} a_0 + b_0 \ln a &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta, \\ a_n a^n + b_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n \geq 1, \\ c_n a^n + d_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n \geq 1, \\ a_0 + b_0 \ln b &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)d\theta, \\ a_n b^n + b_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta, \quad n \geq 1, \\ c_n b^n + d_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta, \quad n \geq 1. \end{aligned}$$

**Example.** Solve

$$\begin{aligned} PDE: &\qquad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 3; \\ BC: &\qquad u(1,\theta) = 2\cos^2(\theta) - 1, u(3,\theta) = \sin(3\theta). \end{aligned}$$

Solution: $a = 1,\ b = 3,\ f(\theta) = \cos(2\theta),\ g(\theta) = \sin(3\theta)$.
**Method 1**:

$$\begin{aligned} a_0 + b_0 \ln 1 &= \frac{1}{\pi}\int_0^{2\pi} \cos(2\theta)d\theta = 0, \\ a_n 1^n + b_n 1^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \cos(2\theta)\cos(n\theta)d\theta = \begin{cases} 1, & \text{if } n = 2; \\ 0, & \text{if } n \neq 2. \end{cases} \\ c_n 1^n + d_n 1^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \cos(2\theta)\sin(n\theta)d\theta = 0, \quad n \geq 1, \\ a_0 + b_0 \ln 3 &= \frac{1}{\pi}\int_0^{2\pi} \sin(3\theta)d\theta = 0, \\ a_n 3^n + b_n 3^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \sin(3\theta)\cos(n\theta)d\theta = 0, \quad n \geq 1, \\ c_n 3^n + d_n 3^{-n} &= \frac{1}{\pi}\int_0^{2\pi} \sin(3\theta)\sin(n\theta)d\theta = \begin{cases} 1, & \text{if } n = 3; \\ 0, & \text{if } n \neq 3. \end{cases} \end{aligned}$$

**Method 2**: We start from the boundary conditions directly.

$$u(r,\theta) = \frac{a_0 + b_0 \ln r}{2} + \sum_{n=1}^{\infty}[(a_n r^n + b_n r^{-n})\cos(n\theta) + (c_n r^n + d_n r^{-n})\sin(n\theta)] \Rightarrow$$

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[(a_n + b_n)\cos(n\theta) + (c_n + d_n)\sin(n\theta)] = \cos(2\theta) \Rightarrow$$

$$\frac{a_0}{2} = 0, \quad a_2 + b_2 = 1, \quad a_n + b_n = 0\ (n \neq 2), \quad c_n + d_n = 0\ (n \geq 1). \tag{47}$$

Similarly,

$$u(3,\theta) = \frac{a_0 + b_0 \ln 3}{2} + \sum_{n=1}^{\infty}[(a_n 3^n + b_n 3^{-n})\cos(n\theta) + (c_n 3^n + d_n 3^{-n})\sin(n\theta)] = \sin(3\theta) \Rightarrow$$

$$\frac{a_0 + b_0 \ln 3}{2} = 0, \quad a_n 3^n + b_n 3^{-n} = 0\ (n \neq 1), \quad c_3 3^3 + d_3 3^{-3} = 1, \quad c_n 3^n + d_n 3^{-n} = 0\ (n \neq 3). \tag{48}$$

By (47) and (48), $a_2 = -\frac{1}{80}$, other $a_n = 0$; $b_2 = \frac{81}{80}$, other $b_n = 0$; $c_3 = \frac{27}{728}$, other $c_n = 0$; $d_3 = -\frac{27}{728}$, other $d_n = 0$. Thus

$$u(r,\theta) = \left(-\frac{1}{80}r^2 + \frac{81}{80}r^{-2}\right)\cos(2\theta) + \left(\frac{27}{728}r^3 - \frac{27}{728}r^{-3}\right)\sin(3\theta).$$

# 5 Sturm-Liouville Problems

Every linear second-order ordinary differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, a_0(x) \neq 0$$

can be written in divergence form,

$$[p(x)y']' + s(x)y = 0,$$

where

$$p(x) = e^{\int a_1(x)/a_0(x)dx}, \quad s(x) = \frac{a_2(x)p(x)}{a_0(x)},$$

$p(x)$ is called an integrating factor of the differential eqn.

Lengendre's equation can be written as:

$$[(1 - x^2)y']' + n(n + 1)y = 0.$$

Bessel's equation $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0$ can be written as:

$$(xy')' + \left(x - \frac{\nu^2}{x}\right)y = 0.$$

## 5.1 Regular and Periodic Problems

### 5.1.1 General Theory

In Chapter 4 we solved several initial and boundary value problems. While solving these equations we used the method separation of variables which reduces the problem to one of the following types of *Sturm-Liouville problems.*

**Definition 9.** *A regular Sturm-Liouville Problem is the problem which consists in finding the nonzero solutions on $[a, b]$ of the second-order ordinary differential equation (**Sturm-Liouville equation**):*

$$[p(x)y']' + [\lambda r(x) - q(x)]y = 0 \tag{49}$$

*with the boundary conditions*

$$k_1y(a) + k_2y'(a) = 0, (k_1, k_2) \neq (0, 0) \quad l_1y(b) + l_2y'(b) = 0, (l_1, l_2) \neq (0, 0). \tag{50}$$

*where $p$, $q$, $r$, $p'$ are real-valued and continuous on the interval and $p(x) > 0$ and $r(x) > 0$ on $[a, b]$.*

*The values of $\lambda$ for which there exist nonzero solutions (non-trivial solution) are called* ***eigenvalues****, and the corresponding nonzero solutions are called* ***eigenfunctions****. The function $r(x)$ will be called the* ***weight function****.*

**Remark 1.** *$y(x) = 0$ is a solution of the equation, which is called trivial solution.*

**Weight function $r(x)$:**

**Example 58.** *Find $r(x)$ from the following SLP:*

$$y'' + a(x)y' + \lambda b(x)y = 0, \quad c < x < d.$$

**Solution:** By using the integrating factor we get

$$r(x) = b(x)e^{\int a(x)dx}.$$

**Eigenvalues and eigenfunctions.**

**Example 59.** *Find the eigenvalues $\lambda_n$ and the eigenfunctions $y_n(x)$:*

*(a)* $y'' + \lambda y = 0, \quad y(0) = 0, y(L) = 0.$

***Solution:*** $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, ...$

*(b)* $y'' + \lambda y = 0, \quad y(0) = 0, y'(L) = 0.$

***Solution:*** $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \quad y_n(x) = B_n \sin\left(\frac{(2n+1)\pi x}{2L}\right), \quad n = 0, 1, 2, ...$

*(c)* $y'' + \lambda y = 0, \quad y'(0) = 0, y(L) = 0.$

***Solution:*** $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \quad y_n(x) = A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right), \quad n = 0, 1, 2, ...$

*(d)* $y'' + \lambda y = 0, \quad y'(0) = 0, y'(L) = 0.$

***Solution:*** $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, ...$

Proof. We only give the proof of (b). The proof for (a), (c), (d) are similar.

The indicial equation is $I^2 + \lambda = 0$, $I = \pm\sqrt{-\lambda}$.

(i) When $\lambda < 0$. Set $\lambda = -\nu^2$. Then $I = \pm\nu$. The solution is $y = ce^{\nu x} + de^{-\nu x}$. The two initial conditions $y(0) = 0, y'(L) = 0$ imply that

$$\begin{cases} c + d = 0, \\ c\nu e^{\nu L} - d\nu e^{-\nu L} = 0. \end{cases}$$

We imply that $c = d = 0$, and so $y \equiv 0$, which is not an eigenfunction.

(ii) When $\lambda = 0$. Then $y'' = 0$, hence $y = cx + d$. Combining this with the two initial conditions we also get $c = d = 0$, and so, $y = 0$.

(iii) When $\lambda > 0$. Set $\lambda = \nu^2$. Then $I = \pm\nu i$. The solution is $y = c\cos\nu x + d\sin\nu x$. The two initial conditions $y(0) = 0, y'(L) = 0$ imply that

$$\begin{cases} c = 0, \\ -c\nu\sin\nu L + d\nu\cos\nu L = 0. \end{cases}$$

We imply that $c = 0$, $\nu L = (n + \frac{1}{2})\pi$ $(n = 0, \pm1, \pm2, ...)$. Hence the eigenfunctions are

$$y_n(x) = a_n \sin\left(\frac{2n+1}{2L}\pi x\right), \quad n = 0, 1, 2, ...$$

and the eigenvalues are

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \quad n = 0, 1, 2, ....$$

**Example 60.** *Consider the Sturm-Liouville problem*

$$y'' + 2y' + \lambda y = 0, y(0) + y'(0) = 0, y(1) = 0.$$

*(a) Find all eigenvalues $\lambda_n$ and corresponding eigenfunctions $y_n(x)$.*

*(b) Determine the weight function.*

Solution: (a) The indicial (characteristic) eqn is: $I^2 + 2I + \lambda = 0$, $I = -1 \pm \sqrt{1-\lambda}$.

(i) $\lambda = 1$. Then $I = -1$ is the only root.

$$y(x) = (A + Bx)e^{-x}.$$

From the boundary conditions, $A = 0$ and $B = 0$. Thus, $y = 0$, $\lambda = 0$ is not an eigenvalue.

(ii) $\lambda < 1$. Let $\lambda = 1 - \omega^2$, $I = -1 \pm \omega$,

$$y(x) = Ae^{(-1+\omega)x} + Be^{(-1-\omega)x}.$$

From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda < 1$ is not an eigenvalue.

(iii) $\lambda > 1$. Now let $\lambda = 1 + \omega^2$, $I = -1 \pm i\omega$,

$$y(x) = e^{-x}(A\cos\omega x + B\sin\omega x).$$

We have

$$y(0) = A, y'(0) = -A + \omega B.$$

By $y(0) + y'(0) = 0$ we imply that $B = 0$ and $y(x) = e^{-x} A \cos \omega x$. Then by $y(1) = 0$ we have

$$\cos \omega = 0 \Rightarrow \omega = \frac{(2n+1)\pi}{2}, \Rightarrow \lambda_n = 1 + \frac{(2n+1)^2\pi^2}{4}.$$

$$y_n(x) = a_n e^{-x} \cos\left(\frac{(2n+1)\pi x}{2}\right).$$

(b) The integrating factor is

$$p(x) = e^{\int 2dx} = e^{2x}.$$

Multiply two sides of the equation by $p(x)$ we imply that

$$e^{2x}y'' + 2e^{2x}y' + \lambda e^{2x}y = 0 \Rightarrow [e^{2x}y']' + \lambda e^{2x}y = 0.$$

Thus the weight function is $e^{2x}$.

**Periodic Sturm-Liouville problem:** If $p(a) = p(b) \neq 0$, then (50) can be replaced by "**periodic boundary condition**"

$$y(a) = y(b), \quad y'(a) = y'(b).$$

**Example 61.** *Solve* $y'' + \lambda y = 0$ *with periodic boundary conditions* $y(0) = y(L)$ *and* $y'(0) = y'(L)$.

**Solution:** The eigenvalues are $\lambda_n = \left(\frac{2n\pi}{L}\right)^2$ with corresponding eigenfunctions $y_n(x) = A_n \cos(\frac{2n\pi x}{L}) + B_n \sin(\frac{2nx}{L})$ for $n = 0, 1, 2, \ldots$.

**Orthogonality**.

**Definition 10.** *The inner product of two functions* $X_1$ *and* $X_2$ *on* $[a, b]$ *is defined to be*

$$< X_1, X_2 >= \int_a^b X_1(x)\overline{X_2(x)}\, dx,$$

*where* $\overline{X_2(x)}$ *is the complex conjugate of* $X_2(x)$. *If* $< X_1, X_2 >= 0$ *on* $[a, b]$, *then* $X_1$ *and* $X_2$ *are* ***orthogonal*** *on* $[a, b]$. *If* $< X_1, X_1 >= 1$, *then* $X_1(x)$ *is said to be normalized.*

*If*

$$\int_a^b r(x)X_1(x)X_2(x)dx = 0,$$

*then we say that $X_1$ and $X_2$ are* orthogonal *with respect to the weight function $r$.*

**Proposition 2.** *Suppose that $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of the Sturm-Liouville equation* (49) *with corresponding eigenvectors $y_1$ and $y_2$. Then $y_1$ and $y_2$ are orthogonal with respect to the weight function $r$.*

**Theorem 21.**

1. *The eigenvalues of Sturm-Liouville problem are real;*
2. *Every Sturm-Liouville problem has an infinite number of eigenvalues $\lambda_n$ with $\lambda_1 < \lambda_2 < \ldots$ and $\lim_{n\to\infty} \lambda_n = \infty$;*
3. *Every eigenvalue is simple, i.e., admits only one linearly independent eigenfunction, except in the case of periodic boundary conditions, where an eigenvalue may have two linearly independent eigenfunctions;*
4. *Every eigenfunction to a simple eigenvalue is real, except for complex constant multiple. If $\lambda$ is not simple, to every pair of complex linearly independent eigenfunctions corresponding to $\lambda$, there is a pair of real linearly independent eigenfunctions corresponding to $\lambda$.*

**Theorem 22.** *(Pinsky) Let $y_j$ $(j \geq 1)$ be the eigenfunctions of a Sturm-Liouville problem on $[a, b]$ corresponding to the eigenvalues $\lambda_j$. Let $f$ and $f'$ be piecewise continuous. Then*

$$\sum_{j=1}^{\infty} c_j y_j(x) = \frac{f(x+) + f(x-)}{2},$$

*where the coefficients are given by the formulas*

$$c_j = \frac{\int_a^b f(x)y_j(x)r(x)dx}{\int_a^b y_j^2(x)r(x)dx} = \frac{< f, y_j r >}{< y_j, y_j r >}, \quad j = 1, 2, \ldots.$$

**Example 62.** *(a) Solve the Sturm-Liouville problem:*

$$Y'' + 2Y' + (\lambda + 1)Y = 0, \quad Y(0) = 0, Y(2) = 0.$$

*(b) If*

$$e^{-x} = \sum_{n=1}^{\infty} c_n Y_n(x),$$

*where $Y_n(x)$ are eigenfunctions, find $c_n$.*

Solution: (a) The characteristic eqn is: $c^2 + 2c + (\lambda + 1) = 0$, $c = -1 \pm \sqrt{-\lambda}$.

(i) $\lambda = 0$. Then $c = -1$. $Y(x) = (A + Bx)e^{-x}$. From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda = 0$ is not an eigenvalue.

(ii) Now let $\lambda = -\omega^2$, $c = -1 \pm \omega$, $Y(x) = Ae^{(-1+\omega)x} + Be^{(-1-\omega)x}$. From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda = 0$ is not an eigenvalue.

(iii) Now let $\lambda = \omega^2$, $c = -1 \pm i\omega$, $Y(x) = e^{-x}(A\cos\omega x + B\sin\omega x)$. From the boundary conditions, $A = 0$ and $\sin 2\omega = 0$. Thus $\omega = \frac{n\pi}{2}$, and

$$Y_n(x) = e^{-x}\sin\frac{n\pi x}{2}.$$

(b) Note that $r(x) = e^{2x}$,

$$c_n = \frac{\int_a^b f(x)Y_n(x)r(x)dx}{\int_a^b Y_n^2(x)r(x)dx} = \frac{\int_0^2 e^{-x}e^{-x}\sin\frac{n\pi x}{2}e^{2x}dx}{\int_0^2\left(e^{-x}\sin\frac{n\pi x}{2}\right)^2 e^{2x}dx} = \frac{\int_0^2 \sin\frac{n\pi x}{2}dx}{\int_0^2 \frac{1}{2}[1-\cos(n\pi x)]dx}.$$

## 5.2 Singular Problems

### 5.2.1 Bessel's equation

Consider the Bessel's equation

$$x^2y''(x) + xy'(x) + (\lambda^2x^2 - \nu^2)y(x) = 0, \quad x \in [0, b]$$

under

$$BCs : |y(0)| < \infty, \lim_{x\to 0^+} xy'(x) = 0, y(b) = 0.$$

**Example 63.** *Bessel's equation can be written as a Sturm-Liouville equation:*

$$(xy')' + \left(-\frac{\nu^2}{x} + \lambda^2 x\right)y = 0.$$

We now discuss properties of Bessel's theory, not only because of their great practical importance but also as a model case for showing how properties of functions can be discovered from their series.

**Bessel Identities:**

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x), \quad \nu > 0; \tag{51}$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x), \quad \nu \geq 0; \tag{52}$$

Proof. Recall that

$$J_\nu(x) = x^\nu \sum_{m=0}^\infty \frac{(-1)^m}{2^{2m+\nu}m!\Gamma(m+\nu+1)}x^{2m}. \tag{53}$$

Multiplying the two sides by $x^\nu$ we get

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu} m! \Gamma(m+\nu+1)} x^{2m+2\nu}.$$

We differentiate this, by using recursive relation of Gamma function,

$$(x^\nu J_\nu(x))' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\nu)}{2^{2m+\nu} m! \Gamma(m+\nu+1)} x^{2m+2\nu-1} = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu-1} m! \Gamma(m+\nu)} x^{2m}.$$

This proves (51).

Similarly, multiplying the two sides of (53) by $x^{-\nu}$, then calculate derivative, then replace $m$ by $m+1$, we can get (52).

By substitution $t = ax$, we have

**Bessel Identities:**

$$\frac{1}{a}\frac{d}{dx}\left[x^\nu J_\nu(ax)\right] = x^\nu J_{\nu-1}(ax), \quad \nu > 0, a \neq 0; \tag{54}$$

$$\frac{1}{a}\frac{d}{dx}\left[x^{-\nu} J_\nu(ax)\right] = -x^{-\nu} J_{\nu+1}(ax), \quad \nu \geq 0, a \neq 0. \tag{55}$$

**Example 64.** $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x); \quad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x), \quad \nu \geq 0.$

Proof. By (51) and (52),

$$\begin{aligned} \nu x^{\nu-1} J_\nu(x) + x^\nu J'_\nu &= x^\nu J_{\nu-1}, \\ -\nu x^{-\nu-1} J_\nu(x) + x^{-\nu} J'_\nu &= -x^{-\nu} J_{\nu+1}. \end{aligned}$$

**Propositions:**

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x; \tag{56}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{57}$$

Proof. Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We have

$$\Gamma(m + \frac{3}{2}) = 2^{-(m+1)}(2m+1)(2m-1)\cdots 3 \cdot 1\sqrt{\pi}.$$

Now for $\nu = \frac{1}{2}$,

$$\begin{aligned} J_{1/2}(x) &= \sqrt{x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+1/2}m!\Gamma(m+1/2+1)} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!\Gamma(m+3/2)} \\ &= \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!2^{-(m+1)}(2m+1)(2m-1)\cdots 3\cdot 1\sqrt{\pi}} \\ &= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} \\ &= \sqrt{\frac{2}{\pi x}}\sin x. \end{aligned}$$

This proves (56).

Now by (51),

$$x^{1/2}J_{-1/2}(x) = \left[\sqrt{x}J_{1/2}(x)\right]' = \sqrt{\frac{2}{\pi}}\cos x.$$

This proves (57).

**Example 65.** *Integrate* $\int x^3 J_0(kx)dx$.

**Example 66.** *Integrate*

$$I = \int_1^2 x^{-3}J_4(x)\,dx.$$

Solution. By (52),

$$I = -\int_1^2 \left[x^{-3}J_3(x)\right]'\,dx = -2^{-3}J_3(2) - J_3(1).$$

Note that

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x), \quad J_2(x) = \frac{2}{x}J_1(x) - J_0(x).$$

We imply that

$$J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x). \tag{58}$$

From the table we can find the value $I = 0.0038$.

**Example 67.** *Represent* $J_4(x)$ *by using* $J_0(x)$ *and* $J_1(x)$.

**Solution:** From

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x)$$

we imply that

$$\begin{aligned} J_0(x)+J_2(x) &= \frac{2}{x}J_1(x), \\ J_1(x)+J_3(x) &= \frac{4}{x}J_2(x), \\ J_2(x)+J_4(x) &= \frac{6}{x}J_3(x), \\ J_3(x)+J_5(x) &= \frac{8}{x}J_4(x). \end{aligned}$$

Hence

$$\begin{aligned} J_2(x) &= \frac{2}{x}J_1(x)-J_0(x), \\ J_3(x) &= \left(\frac{8}{x^2}-1\right)J_1(x)-\frac{4}{x}J_0(x), \\ J_4(x) &= \left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)-\left(\frac{24}{x^2}-1\right)J_0(x). \end{aligned}$$

**Theorem 23.** *The Bessel functions of the first kind, $J_n(x)$, satisfy the following orthogonality relations:*

$$(J_n(\lambda_{nm}x), J_n(\lambda_{nk}x)) := \int_0^R xJ_n(\lambda_{nm}x)J_n(\lambda_{nk}x)\,dx = \begin{cases} 0, & \text{if } m \neq k; \\ \frac{R^2}{2}J_{n+1}^2(\lambda_{nk}R), & \text{if } m = k. \end{cases}$$

*where $\lambda_{nk} = \frac{\alpha_{nk}}{R}$ and $\alpha_{nk}$ is the kth strictly positive zero of $J_n(x)$.*

**Example 68.**

$$\int_0^4 xJ_5^2(\lambda_{5k}x)\,dx = \frac{R^2}{2}J_{n+1}^2(\lambda_{nk}R) = 8J_6^2(\alpha_{5k}).$$

### 5.2.2 The Vibrating Membrane

By using polar coordinates: $x = r\cos\theta, \quad y = r\sin\theta$, Then

$$u_{xx}+u_{yy} = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}.$$

We consider a circular vibrating membrane of radius R, like a drumhead. We assume that the membrane is well-stretched and its boundary is fixed. The position of the membrane is governed by the wave partial differential equation:

PDE: $u_{tt} = c^2\nabla^2u = c^2(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}), 0 < r < R, t > 0, 0 \leq \theta \leq 2\pi;$

BC: $u(R,\theta,t) = 0$ for all $\theta$ and $t$;

ICs: $u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = g(r,\theta).$

We also impose the physical assumption that $u(r,\theta,t)$ remains bounded for $0 \le r \le R$ and $t > 0$.

We restrict the problem to the simple case where the functions $f$ and $g$ depend only on $r$. Consequently, it is reasonable to suppose that the solution is a function of $r$ and $t$ only.

**Theorem 24.** *Consider*
$PDE: u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r), 0 < r < R, t > 0;$
$BC: u(R,t) = 0$ *for all* $t$*;*
$ICs: u(r,0) = f(r), u_t(r,0) = g(r).$
*The solution is*

$$u(r,t) = \sum_{m=1}^{\infty} \left( a_m \cos(\frac{c\alpha_m}{R}t) + b_m \sin(\frac{c\alpha_m}{R}t) \right) J_0(\frac{\alpha_m}{R}r),$$

$$a_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R rf(r)J_0(\frac{\alpha_m}{R}r)\, dr, \quad m = 1, 2, ...$$

$$b_m = \frac{2}{c\alpha_m R J_1^2(\alpha_m)} \int_0^R rg(r)J_0(\frac{\alpha_m}{R}r)\, dr, \quad m = 1, 2, ...$$

*where* $\alpha_m, m = 1, 2, ...$ *are the zeros of* $J_0(r)$.

We proceed by the familiar three steps.

Step 1. Separation of variables. Let $u(r,t) = W(r)T(t)$.

$$\frac{T''(t)}{c^2T(t)} = \frac{W''(r)}{W(r)} + \frac{1}{r}\frac{W'(r)}{W(r)} = -k^2, \Rightarrow$$

$$T''(t) + \lambda^2 T(t) = 0, \quad \lambda = ck, \Rightarrow T(t) = A\cos(\lambda t) + B\sin(\lambda t),$$

and

$$r^2W''(r) + rW'(r) + (kr)^2W(r) = 0,$$

which is Bessel's equation with order 0, and then

$$W(r) = DJ_0(kr) + EY_0(kr).$$

Step 2. By the physical hypothesis, $u(0,t) = W(0)T(t)$ is bounded, we must take $E = 0$, so that, $W(r) = DJ_0(kr)$. From the boundary condition $u(R,t) = 0$ we see that $W(R) = 0, \Rightarrow J_0(kR) = 0, \Rightarrow k_mR = \alpha_m$ which are positive zeros of $J_0(r)$, i.e., $k_m = \frac{\alpha_m}{R}$. Thus

$$\lambda_m = ck_m = \frac{c\alpha_m}{R}.$$

$$u_m(r,t) = W_m(r)T_m(t) = \left(a_m\cos(\frac{c\alpha_m}{R}t) + b_m\sin(\frac{c\alpha_m}{R}t)\right)J_0(\frac{\alpha_m}{R}r).$$

Step 3.

$$u(r,t) = \sum_{m=1}^{\infty} u_m(r,t) = \sum_{m=1}^{\infty}\left(a_m\cos(\frac{c\alpha_m}{R}t) + b_m\sin(\frac{c\alpha_m}{R}t)\right)J_0(\frac{\alpha_m}{R}r).$$

$$u(r,0) = \sum_{m=1}^{\infty} a_m J_0(\frac{\alpha_m}{R}r) = f(r);$$

$$u_t(r,0) = \sum_{m=1}^{\infty} b_m \frac{c\alpha_m}{R} J_0(\frac{\alpha_m}{R}r) = g(r).$$

By the orthogonality of $J_0$ :

$$\int_0^R xJ_0(\frac{\alpha_m}{R}x)J_0(\frac{\alpha_n}{R}x)\,dx = \begin{cases} 0, & \text{if } m \neq n; \\ \frac{R^2}{2}J_1^2(\alpha_n), & \text{if } m = n. \end{cases}$$

We imply that

$$a_m = \frac{2}{R^2J_1^2(\alpha_m)}\int_0^R rf(r)J_0(\frac{\alpha_m}{R}r)\,dr, \quad m = 1, 2, ...$$

$$b_m = \frac{2}{c\alpha_m RJ_1^2(\alpha_m)}\int_0^R rg(r)J_0(\frac{\alpha_m}{R}r)\,dr, \quad m = 1, 2, ...$$

**Example 69.** *Solve PDE* : $u_{tt} = 4(u_{rr} + \frac{1}{r}u_r), 0 < r < 3, t > 0;$

*BC* : $u(3,t) = 0$ *for all* $t$*;*

*ICs* : $u(r,0) = 0, u_t(r,0) = 5J_0(\frac{\alpha_4}{3}r).$

Solution: $c = 2,\ R = 3,\ f(r) = 0,\ g(r) = 5J_0(\frac{\alpha_4}{3}r)$. Thus $a_m = 0$ for all $m$.

$$u(r,t) = \sum_{m=1}^{\infty} b_m\sin(\frac{2\alpha_m}{3}t)J_0(\frac{\alpha_m}{3}r),$$

$$u_t(r,0) = \sum_{m=1}^{\infty} b_m\frac{2\alpha_m}{3}J_0(\frac{\alpha_m}{3}r) = 5J_0(\frac{\alpha_4}{3}r) = g(r) \Rightarrow$$

$b_4\frac{2\alpha_4}{3} = 5, \Rightarrow b_4 = \frac{15}{2\alpha_4}$, other $b_m = 0$.

$$u(r,t) = \frac{15}{2\alpha_4}\sin(\frac{2\alpha_4}{3}t)J_0(\frac{\alpha_4}{3}r).$$

**Example 70.** *Solve PDE* : $u_{tt} = u_{rr} + \frac{1}{r}u_r, 0 < r < 1, t > 0;$

*BC* : $u(1,t) = 0$ *for all* $t$*;*

*ICs* : $u(r,0) = 1 - r^4, u_t(r,0) = 0.$

Solution: $c = 1$, $R = 1$, $f(r) = 1 - r^4$, $g(r) = 0$. Thus $b_m = 0$ for all $m$.

$$u(r,t) = \sum_{m=1}^{\infty} a_m \cos(\alpha_m t) J_0(\alpha_m r),$$

$$\begin{aligned} a_m &= \frac{2}{J_1^2(\alpha_m)} \int_0^1 r(1 - r^4) J_0(\alpha_m r)\, dr \\ &= \frac{2}{J_1^2(\alpha_m)} \left( \int_0^1 r J_0(\alpha_m r)\, dr - \int_0^1 r^5 J_0(\alpha_m r)\, dr \right) \\ &= \frac{8\alpha_m J_2(\alpha_m) - 16 J_3(\alpha_m)}{\alpha_m^3 J_1^2(\alpha_m)}. \end{aligned}$$

$$u(r,t) = \sum_{m=1}^{\infty} \frac{8\alpha_m J_2(\alpha_m) - 16 J_3(\alpha_m)}{\alpha_m^3 J_1^2(\alpha_m)} \cos(\alpha_m t) J_0(\alpha_m r).$$

# 6 Fourier Transform

## 6.1 Fundamental Properties

**Definition 11.** *The Fourier transform of $f$:*

$$\mathcal{F}(f) := \hat{f}(\lambda) := \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx \tag{59}$$

*and the following* ***inverse Fourier transform*** *of $\hat{f}(\lambda)$:*

$$\mathcal{F}^{-1}(\hat{f}) := f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{-i\lambda x}d\lambda. \tag{60}$$

Remark. The transform $\hat{f}(\lambda)$ can be considered as spectral density, which measures the intensity of $f(x)$ in the frequency domain.

Remark. The unitary forms of the Fourier transform are:

$$\mathcal{F}(f) := \hat{f}(\lambda) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx, \quad \mathcal{F}^{-1}(\hat{f}) := f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{-i\lambda x}d\lambda.$$

**Existence:** If $f$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |f(x)|dx < \infty$, then $\mathcal{F}(f)$ exists. Similarly, if $\hat{f}$ is absolutely integrable, then $\mathcal{F}^{-1}(\hat{f})$ exists.

**Example 71.** *Find the Fourier transform of the function*

$$f(x) = \begin{cases} k, & \text{if } 0 < x < a; \\ 0, & \text{otherwise,} \end{cases}$$

Solution: From (59) by integration,

$$\begin{aligned} \mathcal{F}(f) = \hat{f}(\lambda) &= \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx \\ &= \int_{0}^{a} ke^{i\lambda x}dx = \frac{k(e^{ia\lambda}-1)}{i\lambda}. \end{aligned}$$

**Example 72.** *Show that*

$$\mathcal{F}(u(x-a) - u(x-b)) = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}.$$

Proof: From (59) by integration,

$$\begin{aligned} \mathcal{F}(u(x-a) - u(x-b)) &= \int_{-\infty}^{\infty} (u(x-a) - u(x-b))e^{i\lambda x}dx \\ &= \int_{a}^{b} e^{i\lambda x}dx = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}. \end{aligned}$$

**Example 73.** *Show that*

$$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\lambda^2}.$$

Proof: From (59) by integration,

$$\begin{aligned}
\mathcal{F}(e^{-|x|}) &= \int_{-\infty}^{\infty} e^{-|x|}e^{i\lambda x}dx \\
&= \int_{-\infty}^{0} e^{x}e^{i\lambda x}dx + \int_{0}^{\infty} e^{-x}e^{i\lambda x}dx = \frac{1}{1+\lambda i}e^{(1+i\lambda)x}|_{-\infty}^{0} + \frac{1}{-1+\lambda i}e^{(-1+i\lambda)x}|_{0}^{\infty} \\
&= \frac{1}{1+i\lambda} - \frac{1}{-1+i\lambda} = \frac{2}{1+\lambda^2}.
\end{aligned}$$

Remark. $\lim_{x\to-\infty} e^{(1+i\lambda)x} = 0 = \lim_{x\to\infty} e^{(-1+i\lambda)x}$.

**Example 74.** *Show that*

$$\mathcal{F}(\delta_a) = e^{i\lambda a}.$$

Proof: Note that

$$\delta_a(f) = \int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a).$$

$$\mathcal{F}(\delta_a) = \int_{-\infty}^{\infty} \delta(x-a)e^{i\lambda x}dx = e^{i\lambda x}|_{x=a} = e^{i\lambda a}.$$

**Linearity**

The Fourier transform is a linear operation, that is, the following proposition holds.

**Proposition** ( Linearity of the Fourier transform) For constants $a$ and $b$, and two functions $g(x)$ and $h(x)$, we have

$$\mathcal{F}(ag+bh) = a\mathcal{F}(g) + b\mathcal{F}(h).$$

By using (59) we can easily prove it.

**Theorem 25.** *If $\mathcal{F}(f) := \hat{f}(\lambda)$, then for any real number $\alpha$,*

$$\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|}\hat{f}\left(\frac{\lambda}{\alpha}\right), \quad \mathcal{F}^{-1}\left\{\hat{f}\left(\frac{\lambda}{\alpha}\right)\right\} = |\alpha|f(\alpha x).$$

Proof. It's easy to get this by substitution $y = \alpha x$.

**Example 75.** *For $t > 0$, find*

$$\mathcal{F}(e^{-t|x|}).$$

Solution: Let $f(x) = e^{-|x|}$. Note that

$$\mathcal{F}(e^{-|x|}) = \mathcal{F}(f(x)) = \hat{f}(\lambda) = \frac{2}{1+\lambda^2}.$$

By the above theorem,

$$\begin{aligned}\mathcal{F}(e^{-t|x|}) = \mathcal{F}(f(tx)) &= \frac{1}{t}\hat{f}\left(\frac{\lambda}{t}\right) \\ &= \frac{1}{t}\frac{2}{1+\left(\frac{\lambda}{t}\right)^2} = \frac{2t}{t^2+\lambda^2}.\end{aligned}$$

**Example 76.** *For $t > 0$, $\mathcal{F}^{-1}\left\{\frac{2t}{t^2+\lambda^2}\right\} = e^{-t|x|}$.*

**Example 77.** *By the definition, $\mathcal{F}^{-1}\left\{e^{-|\lambda|}\right\} = \frac{1}{\pi(1+x^2)}$.*

**Shifting theorem**

**Theorem 26.** *Suppose that $f$ is absolutely integrable and $\mathcal{F}(f) := \hat{f}(\lambda)$. Then for any real number $a$,*

- *The first Shifting Theorem:*

$$\mathcal{F}\left\{e^{iax}f(x)\right\} = \hat{f}(\lambda+a), \quad \mathcal{F}^{-1}\left\{\hat{f}(\lambda+a)\right\} = e^{iax}\mathcal{F}^{-1}\left\{\hat{f}(\lambda)\right\}.$$

- *The second Shifting Theorem:*

$$\mathcal{F}\left\{f(x-a)\right\} = e^{ia\lambda}\hat{f}(\lambda), \quad \mathcal{F}^{-1}\left\{e^{ia\lambda}\hat{f}(\lambda)\right\} = f(x-a).$$

Proof. By the definition we will get both of them.

**Example 78.**

$$\mathcal{F}(e^{-|x+a|}) = \frac{2e^{-ia\lambda}}{1+\lambda^2}.$$

$$\mathcal{F}^{-1}\left\{\frac{2}{\lambda^2-4\lambda+5}\right\} = e^{-2ix-|x|}.$$

$$\mathcal{F}^{-1}\left\{\frac{e^{-ia\lambda}}{1+(\lambda-b)^2}\right\} = \frac{1}{2}e^{-bi(x+a)-|x+a|}.$$

$$\mathcal{F}(e^{iax-|x|}) = \frac{2}{1+(\lambda+a)^2}.$$

**Fourier transform and derivatives**

**Theorem 27.** *(Fourier transform of the derivative) Let $f(x)$ be continuous on the x-axis, and let $f(x) \to 0$ as $x \to \infty$. Then*

$$\begin{aligned} \mathcal{F}(f'(x)) &= -i\lambda \mathcal{F}(f(x)), && (61) \\ \mathcal{F}(f''(x)) &= -\lambda^2 \mathcal{F}(f(x)). && (62) \end{aligned}$$

Proof. By (59), we deduce from integration by parts that

$$\begin{aligned} \mathcal{F}(f'(x)) &= \int_{-\infty}^{\infty} f'(x) e^{i\lambda x} dx \\ &= \left[f(x)e^{i\lambda x}\right]_{-\infty}^{\infty} - (i\lambda)\int_{-\infty}^{\infty} f(x)e^{-i\lambda x} dx \\ &= -i\lambda \mathcal{F}(f(x)). \end{aligned}$$

This proves (61). Now by using (61) twice we imply that

$$\mathcal{F}(f''(x)) = -i\lambda\mathcal{F}(f'(x)) = (-i\lambda)^2\mathcal{F}(f(x)) = -\lambda^2\mathcal{F}(f(x)).$$

This proves (62).

**Theorem 28.** *(Derivative of Fourier transform) Suppose that $f(x)$ and $xf(x)$ are absolutely integrable and $\mathcal{F}(f) := \hat{f}(\lambda)$. Then $\hat{f}$ is differentiable and*

$$\mathcal{F}\{xf(x)\} = -i\frac{d\hat{f}(\lambda)}{d\lambda}, \quad \mathcal{F}^{-1}\left\{\frac{d\hat{f}(\lambda)}{d\lambda}\right\} = ixf(x).$$

Proof. Differentiating the two sides

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{i\lambda x} dx$$

will give the result.

**Example 79.** *For $t > 0$,*

$$\mathcal{F}(e^{-tx^2}) = \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\lambda^2/(4t)}, \quad \mathcal{F}^{-1}\left\{e^{-t\lambda^2}\right\} = \frac{1}{2\sqrt{t\pi}}e^{-x^2/(4t)}.$$

Proof. Let $f(x) = e^{-tx^2}$. Then $f'(x) + 2txf(x) = 0, \Rightarrow -i\lambda\hat{f}(\lambda) + 2t(-i)\hat{f}'(\lambda) = 0, \Rightarrow$

$$\frac{d\hat{f}(\lambda)}{\hat{f}(\lambda)} = \frac{\lambda}{2t}d\lambda, \Rightarrow \hat{f}(\lambda) = ce^{-\lambda^2/(4t)}, \quad c = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-tx^2}dx = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

**Example 80.** *Find the Fourier transform of the function $f(x) = xe^{-x^2}$ by using*

$$\mathcal{F}(e^{-tx^2}) = \frac{\sqrt{\pi}}{\sqrt{t}}e^{-\lambda^2/(4t)}.$$

Solution: By the linearity,

$$\begin{aligned}
\mathcal{F}\left(xe^{-x^2}\right) &= \mathcal{F}\left(-\frac{1}{2}[e^{-x^2}]'\right) = -\frac{1}{2}\mathcal{F}\left([e^{-x^2}]'\right) \\
&= -\frac{1}{2}(-i\lambda)\mathcal{F}\left(e^{-x^2}\right) = \frac{i\lambda}{2}\frac{\sqrt{\pi}}{\sqrt{1}}e^{-\lambda^2/4} \\
&= \frac{i\lambda\sqrt{\pi}}{2}e^{-\lambda^2/4}.
\end{aligned}$$

**Example 81.** *Find*

$$\mathcal{F}^{-1}\left\{-\frac{\sqrt{\pi}\lambda}{16}e^{-\lambda^2/16}\right\}.$$

Solution: By the above theorem,

$$\begin{aligned}
\mathcal{F}^{-1}\left\{-\frac{\sqrt{\pi}\lambda}{16}e^{-\lambda^2/16}\right\} &= \mathcal{F}^{-1}\left\{\frac{d}{d\lambda}\left(\frac{\sqrt{\pi}}{2}e^{-\lambda^2/16}\right)\right\} \\
&= ix\mathcal{F}^{-1}\left\{\frac{\sqrt{\pi}}{2}e^{-\lambda^2/16}\right\} \\
&= ixe^{-4x^2}.
\end{aligned}$$

**Convolution** The convolution $f * g$ of functions $f$ and $g$ is defined in Part I:

$$f(x) * g(x) = (f * g)(x) := \int_{-\infty}^{\infty} f(s)g(x-s)ds = \int_{-\infty}^{\infty} f(x-v)g(v)dv.$$

**Theorem 29.** *(Convolution theorem) Let $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable on the x-axis. Then*

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g), \quad (f * g)(x) = \mathcal{F}^{-1}\{\hat{f}(\lambda)\hat{g}(\lambda)\}.$$

**Example 82.** *Let* $f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0, \end{cases}$ $g(x) = \begin{cases} e^{x}, & \text{if } x \leq 0; \\ 0, & \text{if } x > 0. \end{cases}$ *Find* $\mathcal{F}\{f(x)\}$, $\mathcal{F}\{g(x)\}$, $\mathcal{F}\{f(x) * g(x)\}$, $f(x) * g(x)$.

Solution: $\mathcal{F}\{f(x)\} = \frac{1}{1-i\lambda}$, $\mathcal{F}\{g(x)\} = \frac{1}{1+i\lambda}$, $\mathcal{F}\{f(x) * g(x)\} = \frac{1}{1+\lambda^2}$, $f(x) * g(x) = \frac{1}{2}e^{-|x|}$.

## 6.2 Applications

**Theorem 30.** *Consider the initial value problem:*

$$\begin{aligned} PDE: \quad & u_{xx} = u_t, \quad -\infty < x < \infty, t > 0; \\ IC: \quad & u(x,0) = f(x), \quad -\infty < x < \infty. \end{aligned}$$

*The solution is*

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4t}} ds.$$

Proof. The conclusion follows from the Fourier transform of the two sides of the PDE:

$$\mathcal{F}(u_{xx}) = -\lambda^2 \hat{u}(\lambda, t), \quad \mathcal{F}(u_t) = \hat{u}_t(\lambda, t), \quad \hat{u}(\lambda, 0) = \hat{f}(\lambda).$$

**Definition 12.** *The heat kernel is defined to be*

$$K(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

Remark. By using the heat kernel, we have

$$u(x,t) = f(x) * K(x,t).$$

**Example 83.** *Solve the heat equation:*

$$\begin{aligned} \text{PDE:} \quad & u_{xx} = u_t, \quad -\infty < x < \infty, t > 0; \\ \text{IC:} \quad & u(x,0) = \delta(x-a), \quad -\infty < x < \infty. \end{aligned}$$

Solution: By the above theorem,

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \delta(s-a) e^{-\frac{(x-s)^2}{4t}} ds \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-s)^2}{4t}} \Big|_{s=a} \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-a)^2}{4t}}. \end{aligned}$$

**Example 84.** *Solve the heat equation:*

$$\begin{aligned} \text{PDE:} \quad & u_{xx} = u_t, \quad -\infty < x < \infty, t > 0; \\ \text{IC:} \quad & u(x,0) = e^{-3x^2}, \quad -\infty < x < \infty. \end{aligned}$$

Solution: By the above theorem,

$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-3s^2} e^{-\frac{(x-s)^2}{4t}} ds \\
&= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{12t+1}{4t}(s-\frac{x}{12t+1})^2 - \frac{3x^2}{12t+1}} ds \\
&= \frac{e^{-\frac{3x^2}{12t+1}}}{2\sqrt{\pi t}} \sqrt{\frac{4t}{12t+1}} \int_{-\infty}^{\infty} e^{-y^2} dy, \quad y = \sqrt{\frac{12t+1}{4t}}(s - \frac{x}{12t+1}) \\
&= \frac{e^{-\frac{3x^2}{12t+1}}}{2\sqrt{\pi t}} \sqrt{\frac{4t}{12t+1}} \sqrt{\pi} = \frac{e^{-\frac{3x^2}{12t+1}}}{\sqrt{12t+1}}.
\end{aligned}$$