CARLETON UNIVERSITY

FINAL EXAMINATION
APRIL 2016

DURATION: 3 HOURS **SCANTRON FORMS REQUIRED**

Department Name and Course Number: School of Mathematics and Statistics, MATH 3705 A, B, C, D

Course Instructor(s): Dr. E. Hua (Sections A & D), Dr. E. Devdariani (Section B), Dr. S. Melkonian (Section C)

AUTHORIZED MEMORANDA
Non-programmable, non-graphic calculators

1. $\mathcal{L}\{e^{3t}\cos(2t)\} =$
   (a) $\dfrac{s-3}{(s-3)^2+4}$ (b) $\dfrac{s+3}{(s+3)^2+4}$ (c) $\dfrac{s-2}{(s-2)^2+9}$ (d) $\dfrac{s}{(s-3)^2+4}$
   (e) None of these

2. $\mathcal{L}\{t\sin(3t)\} =$
   (a) $\dfrac{-3s}{(s^2+9)^2}$ (b) $\dfrac{3s}{(s^2+9)^2}$ (c) $\dfrac{-6s}{(s^2+9)^2}$ (d) $\dfrac{6s}{(s^2+9)^2}$ (e) None of these

3. $\mathcal{L}^{-1}\left\{\dfrac{-s+7}{s^2+s-2}\right\} =$
   (a) $3e^{t}-2e^{-2t}$ (b) $2e^{-2t}-3e^{t}$ (c) $2e^{t}-3e^{-2t}$ (d) $2e^{-t}-3e^{2t}$ (e) None of these

4. $\mathcal{L}^{-1}\left\{\dfrac{(s+3)e^{-s}}{s^2-6s+13}\right\} =$
   (a) $u(t-1)e^{3t}[\cos(2t)+3\sin(2t)]$ (b) $u(t-1)e^{3(t-1)}\big\{\cos[2(t-1)]+6\sin[2(t-1)]\big\}$
   (c) $u(t-1)e^{3t}[\cos(2t)+6\sin(2t)]$ (d) $u(t-1)e^{3(t-1)}\big\{\cos[2(t-1)]+3\sin[2(t-1)]\big\}$
   (e) None of these

5. If $y(t)$ is the solution of the initial-value problem $y''-2y'+3y=t$, $y(0)=2$, $y'(0)=-1$, then $Y(s)=\mathcal{L}\{y(t)\} =$
   (a) $\dfrac{2s+3+\frac{1}{s^2}}{s^2-2s+3}$ (b) $\dfrac{2s-5+\frac{1}{s^2}}{s^2-2s+3}$ (c) $\dfrac{2s-3+\frac{1}{s^2}}{s^2-2s+3}$ (d) $\dfrac{\frac{1}{s^2}-2s-6}{s^2-2s+3}$
   (e) None of these

6. If $F(s) = \mathcal{L}\{f(t)\} = \dfrac{1}{(s-2)^4}$, then $f(t) =$

   (a) $\frac{1}{3}t^3e^{2t}$ (b) t^3e^{2t} (c) $\frac{1}{6}t^3e^{-2t}$ (d) $\frac{1}{6}t^3e^{2t}$ (e) None of these

7. The general solution of the differential equation $x^2y'' + 7xy' + 13y = 0$, $x \neq 0$, is

   (a) $|x|^{-\frac{7}{2}}\left[c_1\cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2\sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right]$ (b) $e^{-3x}[c_1\cos(2x) + c_2\sin(2x)]$

   (c) $e^{-\frac{7}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$ (d) $x^{-3}\left[c_1\cos(2\ln|x|) + c_2\sin(2\ln|x|)\right]$

   (e) None of these

8. The general solution of the differential equation $16x^2y'' + 8xy' + y = 0$, $x \neq 0$, is

   (a) $|x|^{-\frac{1}{4}}(c_1 + c_2\ln|x|)$ (b) $c_1|x|^{\frac{1}{4}} + c_2|x|^{\frac{1}{4}}$ (c) $c_1|x|^{-\frac{1}{4}} + c_2|x|^{-\frac{1}{4}}$

   (d) $|x|^{\frac{1}{4}}(c_1 + c_2\ln|x|)$ (e) None of these

9. The coefficient recursion relation of the solution $y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$ of the differential equation $xy'' + (x-1)y' - y = 0$ is

   (a) $a_{n+1} = \dfrac{-a_n}{n+3}$ (b) $a_{n+1} = \dfrac{a_n}{n+3}$ (c) $a_{n+1} = \dfrac{a_n}{(n+2)(n+3)}$ (d) $a_{n+1} = \dfrac{-a_n}{n+2}$

   (e) None of these

10. The solution of the coefficient recursion relation $a_{n+1} = \dfrac{-a_n}{(n+1)^2}$, $n \geq 0$, is $a_n =$

    (a) $\dfrac{(-1)^n a_0}{(n!)^2}$ (b) $\dfrac{a_0}{(n!)^2}$ (c) $\dfrac{-a_0}{(n+1)^2}$ (d) $\dfrac{(-1)^n a_0}{(n+1)^2}$ (e) None of these

11. One solution of the differential equation $x^2y'' + (x + x^2)y' + (x-9)y = 0$ has the form

    (a) $\sum_{n=0}^{\infty} a_n x^n$ (b) $\sum_{n=0}^{\infty} a_n x^{n+1}$ (c) $\sum_{n=0}^{\infty} a_n x^{n+2}$ (d) $\sum_{n=0}^{\infty} a_n x^{n+3}$ (e) None of these

12. The general solution of the differential equation $x^2y'' + xy' + (2x^2 - 9)y = 0$, $x > 0$, is

    (a) $c_1J_3(\sqrt{2}\,x) + c_2J_{-3}(\sqrt{2}\,x)$ (b) $c_1J_3(\sqrt{2}\,x) + c_2Y_3(\sqrt{2}\,x)$

    (c) $c_1J_{\sqrt{2}}(3x) + c_2J_{-\sqrt{2}}(3x)$ (d) $c_1J_{\sqrt{2}}(3x) + c_2Y_{\sqrt{2}}(3x)$ (e) None of these

13. At $x = 58$, the Fourier sine series of $f(x) = \left\{\begin{array}{ll} x^2, & 0 \leq x \leq 2 \\ 2, & 2 < x \leq 3 \end{array}\right\}$ on $[0,3]$ converges to

    (a) 0 (b) -3 (c) 3 (d) 1 (e) None of these

14. The Fourier sine series of $f(x) = \left\{ \begin{array}{ll} 1, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{array} \right\}$ on $[0, 2]$ is

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right] \sin\left(\frac{n\pi x}{2}\right)$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - (-1)^n] \sin\left(\frac{n\pi x}{2}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin(n\pi x)$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\pi}[1 - \cos(2n\pi)] \sin(n\pi x)$ (e) None of these

15. The solution of the heat equation $u_{xx} = u_t$, $0 < x < 1$, $t > 0$, which satisfies the boundary conditions $u_x(0, t) = u_x(1, t) = 0$ and the initial condition $u(x, 0) = x$, is $u(x, t) =$

(a) $\displaystyle\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n^2\pi^2} \cos(n\pi x)e^{-n^2\pi^2 t}$ (b) $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi^2} \cos(n\pi x)e^{-n^2\pi^2 t}$

(c) $\displaystyle\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2} \cos(n\pi x)e^{-n^2\pi^2 t}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin(n\pi x)e^{-n^2\pi^2 t}$

(e) None of these

16. The solution of the wave equation $u_{xx} = \frac{1}{9}u_{tt}$, $0 < x < 2$, $t > 0$, which satisfies the boundary conditions $u(0, t) = 0$ and $u(2, t) = 0$ and the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = 3\sin(2\pi x) - 2\sin(3\pi x)$, is

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left[a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right)\right], \text{ where}$$

(a) $b_4 = \frac{1}{2\pi}$, $b_6 = \frac{-2}{9\pi}$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \ge 1$

(b) $b_4 = 3$, $b_6 = -2$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \ge 1$

(c) $b_2 = 3$, $b_3 = -2$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \ge 1$

(d) $a_2 = 3$, $a_3 = -2$, $a_n = 0$ otherwise, $b_n = 0$ for all $n \ge 1$

(e) None of these

17. The solution $u(x, y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangular region $0 < x < 2$, $0 < y < 3$, subject to the boundary conditions $u(0, y) = 0$, $u(2, y) = 0$, $u(x, 0) = 2x - x^2$, $u(x, 3) = 0$, has the form

(a) $\displaystyle u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$

(b) $\displaystyle u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(3-y)}{2}\right] \sin\left(\frac{n\pi x}{2}\right)$

(c) $\displaystyle u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right)$

(d) $\displaystyle u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(2-x)}{3}\right] \sin\left(\frac{n\pi y}{3}\right)$

(e) $u(x, y) = \alpha x + \beta y + \gamma xy + \delta$

18. The solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ within the circle $r = 2$, which satisfies the boundary condition $u(2, \theta) = 2 - 3\sin(2\theta) + 2\cos(3\theta)$, is
$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \Big[a_n \cos(n\theta) + b_n \sin(n\theta)\Big]$, where
(a) $a_0 = 4$, $b_2 = -3$, $a_3 = 2$, $a_n = b_n = 0$ otherwise
(b) $a_0 = 4$, $a_2 = -3$, $b_3 = 2$, $a_n = b_n = 0$ otherwise
(c) $a_0 = 4$, $a_2 = -\frac{3}{4}$, $b_3 = \frac{1}{4}$, $a_n = b_n = 0$ otherwise
(d) $a_0 = 4$, $b_2 = -\frac{3}{4}$, $a_3 = \frac{1}{4}$, $a_n = b_n = 0$ otherwise
(e) None of these

19. The differential equation $xy'' + 3y' - xy + \lambda x^2 y = 0$, when placed in the Sturm-Liouville form $[p(x)y']' - q(x)y + \lambda r(x)y = 0$, has the weight function $r(x) =$
(a) x^3 (b) x^4 (c) x^5 (d) $x^2 e^{3x}$ (e) None of these

20. Given the Bessel identity $\frac{1}{\alpha}\frac{d}{dx}\Big[x^\nu J_\nu(\alpha x)\Big] = x^\nu J_{\nu-1}(\alpha x)$, $\nu \geq 0$, $\alpha > 0$,
$\int_0^3 x^6 J_3(2x)\, dx =$
(a) $3^6 J_4(6) - 3^5 J_5(6)$ (b) $\frac{3^6}{2}\Big[J_4(6) - J_5(6)\Big]$ (c) $\frac{1}{2}\Big[3^6 J_4(6) - 3^5 J_5(6)\Big]$
(d) $3^6 J_3(6)$ (e) None of these

21. The eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0$, $0 < x < 2$, $y'(0) = 0$, $y(2) = 0$, are
(a) $\lambda_n = \frac{n^2\pi^2}{4}$, $y_n = A_n \cos\left(\frac{n\pi x}{2}\right)$, $n \geq 0$
(b) $\lambda_n = \frac{(2n+1)^2\pi^2}{4}$, $y_n = A_n \cos\left[\frac{(2n+1)\pi x}{2}\right]$, $n \geq 0$
(c) $\lambda_n = \frac{(2n+1)^2\pi^2}{4}$, $y_n = B_n \sin\left[\frac{(2n+1)\pi x}{2}\right]$, $n \geq 0$
(d) $\lambda_n = \frac{(2n+1)^2\pi^2}{16}$, $y_n = A_n \cos\left[\frac{(2n+1)\pi x}{4}\right]$, $n \geq 0$
(e) None of these

22. $\mathcal{F}\left\{e^{3ix-|x+2|}\right\} =$
(a) $\frac{2e^{-2i(\lambda-3)}}{1+(\lambda-3)^2}$ (b) $\frac{2e^{2i(\lambda-3)}}{1+(\lambda-3)^2}$ (c) $\frac{2e^{-2i(\lambda+3)}}{1+(\lambda+3)^2}$ (d) $\frac{2e^{2i(\lambda+3)}}{1+(\lambda+3)^2}$
(e) None of these

23. $\mathcal{F}\left\{xe^{-x^2}\right\} =$
(a) $\frac{i\sqrt{\pi}}{2}\lambda e^{-\lambda^2/4}$ (b) $-\frac{i\sqrt{\pi}}{2}\lambda e^{-\lambda^2/4}$ (c) $\frac{i\sqrt{\pi}}{4}\lambda e^{-\lambda^2/4}$ (d) $i\sqrt{\pi}\lambda e^{-\lambda^2/4}$ (e) None of these

24. $\mathcal{F}^{-1}\left\{\dfrac{e^{2i\lambda}}{1+(\lambda-3)^2}\right\} =$

(a) $\dfrac{1}{2}e^{3i(x+2)-|x+2|}$ (b) $\dfrac{1}{2}e^{-3i(x+2)-|x+2|}$ (c) $\dfrac{1}{2}e^{-3i(x-2)-|x-2|}$ (d) $\dfrac{1}{2}e^{3i(x-2)-|x-2|}$

(e) None of these

25. $\mathcal{F}^{-1}\left\{e^{-(\lambda+3)^2}\right\} =$

(a) $\dfrac{1}{2\sqrt{\pi}}e^{-3ix-\frac{x^2}{4}}$ (b) $\dfrac{1}{2\sqrt{\pi}}e^{3ix-\frac{x^2}{4}}$ (c) $e^{3ix}e^{-x^2}$ (d) $e^{-(x+3)^2}$ (e) None of these

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha}F\left(\frac{s}{\alpha}\right), \quad \alpha > 0$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n}F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s), \text{ where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \text{ if } f \text{ is periodic with period } T$$

Summary of Fourier Series

1. The Fourier series of a $2L$-periodic function f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1,$$

where α is any real number. If f is an odd function, then

$$a_n = 0 \text{ and } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

If f is an even function, then

$$b_n = 0 \text{ and } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

2. The Fourier series of a function f defined on $[a, b]$ with $b - a = 2L$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L}\int_a^b f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L}\int_a^b f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

If the $2L$-periodic extension $\tilde{f}$ of f to $\mathbb{R}$ is an odd function, then $a_n = 0$, and if $\tilde{f}$ is an even function, then $b_n = 0$.

3. The Fourier sine series of a function f defined on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

4. The Fourier cosine series of a function f defined on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

Table of Fourier Transforms

$$\mathcal{F}\{f(x)\} = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{i\lambda x}\, dx$$

$$\mathcal{F}^{-1}\{F(\lambda)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}\, d\lambda$$

$$\mathcal{F}\{u(x-a) - u(x-b)\} = \frac{1}{i\lambda}\left(e^{i\lambda b} - e^{i\lambda a}\right),\ a < b$$

$$\mathcal{F}\{u(x+b) - u(x-b)\} = \frac{1}{i\lambda}\left(e^{i\lambda b} - e^{-i\lambda b}\right) = \frac{2}{\lambda}\sin(\lambda b)$$

$$\mathcal{F}\left\{e^{-|x|}\right\} = \frac{2}{1+\lambda^2}$$

$$\mathcal{F}\left\{e^{iax} f(x)\right\} = \widehat{f}(\lambda + a)$$

$$\mathcal{F}\{f(x-a)\} = e^{i\lambda a}\widehat{f}(\lambda)$$

$$\mathcal{F}\{f'(x)\} = -i\lambda\widehat{f}(\lambda)$$

$$\mathcal{F}\{xf(x)\} = -i\frac{d\widehat{f}}{d\lambda}$$

$$\mathcal{F}\left\{e^{-tx^2}\right\} = \sqrt{\frac{\pi}{t}}\, e^{-\frac{\lambda^2}{4t}},\ t > 0$$

$$\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|}\widehat{f}\left(\frac{\lambda}{\alpha}\right),\ \alpha \neq 0$$

$$\mathcal{F}\{(f * g)(x)\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s) g(x-s)\, ds\right\} = \widehat{f}(\lambda)\widehat{g}(\lambda),\ \text{where } \widehat{g}(\lambda) = \mathcal{F}\{g(x)\}$$

$$\mathcal{F}\{\delta(x-a)\} = e^{i\lambda a}$$

Answers

1. a
2. d
3. c
4. d
5. b
6. d
7. d
8. d
9. a
10. a
11. d
12. b
13. b
14. a
15. c
16. a
17. b
18. d
19. b
20. c
21. d
22. c
23. a
24. c
25. b