Test 1
MAT 1320 G Calculus I
Fall 2019

Professor: Mahdi Ammar

NAME ____

Student # ____

DGD (circle yours): DGD 1 (8:00-9:50) DGD 2 (10:00-11:20) DGD 3 (11:30-12:50)

Instructions:

- No calculators are permitted. No notes, books, papers of any kind, or any other aids.
- Print your name and student number on this page.
- Verify that you have 8 questions :
  - Questions 1 to 6 are multiple choice questions. Please write your answers in the corresponding box of the grid below.
  - Questions 7 and 8 require a detailed answer. Please write all of your work in the space provided.
- You can use the backs of the pages as scrap papers.
- Good luck.

Write your answer to each question of the part Multiple-choice Questions in the following boxes:

Q1	Q2	Q3	Q4	Q5	Q6
A	A	B	D	F	D

Do not write below this following table :

MCQ	Q7	Q8	Total
/12	/4	/4	/20

Part I. Multiple-choice Questions ($2 \times 6 = 12$ **marks)**

1. (2 marks) Assume some values of a one-to-one function $y = f(x)$ are given in the following table:

x	1	2	3	4	5
$f(x)$	4	1	5	3	2
$g(x)$	2	4	5	3	1

Which one of the following statements is true?
A. $(f \circ g)(2) = 3,\ f^{-1}(2) = 5$
B. $(f \circ g)(2) = 1,\ f^{-1}(2) = 4$
C. $(f \circ g)(2) = 3,\ f^{-1}(2) = 4$
D. $(f \circ g)(2) = 1,\ f^{-1}(2) = 5$
E. $(f \circ g)(2) = 3,\ f^{-1}(2) = 1$
F. $(f \circ g)(2) = 1,\ f^{-1}(2) = 5$

Proof. (A)
$(f \circ g)(2) = f(g(2)) = f(4) = 3$. Since $f(5) = 2$, $f^{-1}(2) = 5$. □

2. (2 marks) Let f be a function given by

$$f(x) = \begin{cases} ax + 6, & \text{if } x \leq 2; \\ ax^2 - 6, & \text{if } x > 2. \end{cases}$$

For which value of a is $f(x)$ continuous for all real numbers?
A. $a = 6$ B. $a = 2$ C. $a = 1$ D. $a = 4$ E. $a = 9$ F. $a = 8$

Proof. (A)
This function is continuous when $x < 2$ and when $x > 2$. When $x = 2$,
$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} ax + 6 = 2a + 6$ and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - 6 = 4a - 6$. Hence, $2a + 6 = 4a - 6 \Leftrightarrow -2a = -12 \Leftrightarrow a = 6$.

□

3. (2 marks) If $3^x = 5^{x+1}$, then $x =$

A. $\dfrac{\ln(3/5)}{\ln 3}$ B. $\dfrac{\ln 5}{\ln(3/5)}$ C. $\ln\left(\dfrac{3}{5}\right)$ D. $\ln\left(\dfrac{4}{3}\right)$ E. $\dfrac{\ln(3/5)}{\ln 5}$ F. $\dfrac{3}{\ln(3/5)}$

Proof. (B)
Take the natural logarithm on both sides of the equation: $\ln 3^x = \ln 5^{x+1}$.

By the property of the logarithm,

$$x \ln 3 = (x+1)\ln 5 \Leftrightarrow x(\ln 3 - \ln 5) = \ln 5 \Leftrightarrow x = \frac{\ln 5}{\ln(3/5)}.$$

□

4. (2 marks) Find the one-side limit $\displaystyle\lim_{x\to(-3)^-} \frac{x^2+2x-3}{|x+3|} =$

A. -1 B. 1 C. -3 D. 4 E. -5 F. 5

Proof. (D)

$$\lim_{x\to(-3)^-} \frac{x^2+2x-3}{|x+3|} = \lim_{x\to(-3)^-} \frac{(x+3)(x-1)}{-(x+3)} = \lim_{x\to(-3)^-} -(x-1) = 4.$$

□

5. (2 marks) Let $f(x) = e^{-3x}\sin\left(\frac{\pi}{2}x\right)$. The equation of the tangent line to the graph of $f(x)$ at the point $x = 1$ is

A. $y = e^{-3}x$
B. $y = 2e^{-3}x - e^{-3}$
C. $y = -e^{-3}x + 2e^{-3}$
D. $y = -3e^{-3}x + e^{-3}$
E. $y = -2e^{-3}x + 3e^{-3}$
F. $y = -3e^{-3}x + 4e^{-3}$

Proof. (F)

By the product rule and the chain rule,

$$\begin{aligned} f'(x) &= \left(e^{-3x}\right)'\sin\left(\frac{\pi}{2}x\right) + e^{-3x}\left(\sin\left(\frac{\pi}{2}x\right)\right)' \\ &= -3e^{-3x}\sin\left(\frac{\pi}{2}x\right) + e^{-3x}\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) \end{aligned}$$

When$x = 1$, $f'(1) = -3e^{-3}$ and $f(1) = e^{-3}$ The equation of the tangent line of the graph of $f(x)$ at the point $x = 1$ is

$$y = -3e^{-3}(x-1) + e^{-3} = -3e^{-3}x + 4e^{-3}.$$

□

6. (2 marks) Suppose a function $y = f(x)$ is defined implicitly by the equation

$$\frac{x}{y} + 2x + y^2 = 3.$$

Then the derivative of this function at the point $(2, -1)$ is

A. $-\frac{1}{4}$ B. 1 C. $\frac{2}{3}$ D. $\frac{1}{4}$ E. -2 F. $\frac{1}{2}$

Proof. (D)
Taking the derivative on both sides with respect to x, we have
$\frac{y - xy'}{y^2} + 2 + 2yy' = 0$. When $x = 2$, and $y = -1$,

$$2 + y' + 2 - 2y' = 0 \Leftrightarrow y' = \frac{1}{4}$$

□

Part II. Detailed-Answer Questions (8 marks)

7. (4 marks) Consider function $f(x) = \dfrac{x^2 - x - 2}{3x^2 + x - 2}$

(a) (2 marks) Find the domain of the function f and calculate the limit at the points which do not belong to the domain.

Proof. When $3x^2 + x - 2 = 0$, $x = -1$ and $x = \frac{2}{3}$. Then $D_f = \mathbb{R} \setminus \{-1, \frac{2}{3}\}$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x^2 - x - 2}{3x^2 + x - 2} = \lim_{x\to -1} \frac{(x+1)(x-2)}{3(x+1)(x-\frac{2}{3})} = \frac{-1-2}{3(-1-\frac{2}{3})} = \frac{-3}{3(-1-\frac{2}{3})} = \frac{3}{5}.$$

$\lim_{x\to \frac{2}{3}} f(x) = \lim_{x\to \frac{2}{3}} \dfrac{x^2 - x - 2}{3x^2 + x - 2}$ does not exist because

$$\lim_{x\to \left(\frac{2}{3}\right)^-} f(x) = \lim_{x\to \left(\frac{2}{3}\right)^-} \frac{(x-2)}{3(x-\frac{2}{3})} = \frac{(\frac{2}{3}-2)}{0^-} = +\infty$$

$$\lim_{x\to \left(\frac{2}{3}\right)^+} f(x) = \lim_{x\to \left(\frac{2}{3}\right)^+} \frac{(x-2)}{3(x-\frac{2}{3})} = \frac{(\frac{2}{3}-2)}{0^+} = -\infty$$

□

(b) (2 marks) Find the horizontal asymptote of the graph of f, if any. (Answer NONE if there is no horizontal asymptote.)

Proof.

$$\lim_{x\to +\infty} f(x) = \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x^2 - x - 2}{3x^2 + x - 2} = \lim_{x\to -\infty} = \lim_{x\to -\infty} \frac{x^2\left(1 - \frac{1}{x} - \frac{2}{x^2}\right)}{x^2\left(3 + \frac{1}{x} - \frac{2}{x^2}\right)} = \frac{1}{3}.$$

The graph of $f(x)$ has one horizontal asymptote $y = \dfrac{1}{3}$ □

8. (a) (1 mark) Give the definition of the derivative of a function $f(x)$ at a point $x = a$.

Proof. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ □

(b) (3 marks) **Use the definition of the derivative** to find the derivative of the function $f(x) = \sqrt{3x+1}$ at $x = 5$. (Pay attention to your presentation. It counts!)

Proof.

$$\begin{aligned}
f'(5) &= \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(5+h)+1} - \sqrt{3\times 5+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{16+3h} - 4}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{16+3h} - 4\right)\left(\sqrt{16+3h} + 4\right)}{h\left(\sqrt{16+3h} + 4\right)} \\
&= \lim_{h\to 0} \frac{16+3h-16}{h\left(\sqrt{16+3h} + 4\right)} \\
&= \lim_{h\to 0} \frac{3}{\left(\sqrt{16+3h} + 4\right)} \\
&= \frac{3}{\left(\sqrt{16+3\times 0} + 4\right)} \\
&= \frac{3}{8}.
\end{aligned}$$

□