[**2**] **1.** Find the derivative of $f(x) = \dfrac{1}{2x+1}$ using the definition. You may not use any of the rules we saw, only the definition.

solution:

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{2(x+h)+1}-\frac{1}{2x+1}}{h} &= \lim_{h\to 0}\frac{(2x+1)-(2(x+h)+1)}{h\,(2x+1)\,(2(x+h)+1)}\\ &= \lim_{h\to 0}\frac{-2h}{h\,(2x+1)\,(2(x+h)+1)}\\ &= \lim_{h\to 0}\frac{-2}{(2x+1)\,(2(x+h)+1)}\\ &= \frac{-2}{(2x+1)\,(2(x+0)+1)}\\ &= \frac{-2}{(2x+1)^2}\end{aligned}$$

[**1**] **2.** Give the value(s) of a such that the function

$$g(x) = \begin{cases}(x-1)^2 + a & \text{if } x < 2\\ 2^x + ax & \text{if } x \geq 2\end{cases}$$

is continuous everywhere.

solution: The function is continuous when $x \neq 2$ because exponentials, powers, sums, products are all continuous. At $x = 2$ we get:

$$\lim x \to 2^- g(x) = \lim x \to 2^-(x-1)^2 + a = (2-1)^2 + a = 1 + a$$
$$\lim x \to 2^+ g(x) = \lim x \to 2^+ 2^x + ax = 2^2 + a2 = 4 + 2a$$

We set these equal.

$$\begin{aligned}1 + a &= 4 + 2a\\ a &= -3\end{aligned}$$

[**4**] **3.** Find each of the following limits. You may use any technique we have seen so far in the course.

a) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^6+x^2}+x^2}{-2x^3+3x}$

solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x^6+x^2}+x^2}{-2x^3+3x}\times\frac{1/x^3}{1/x^3} &= \lim_{x\to\infty}\frac{\sqrt{1+1/x^4}+1/x}{-2+3/x^2}\\ &= \frac{\sqrt{1+0}+1/x}{-2+0}\\ &= -\frac{1}{2}\end{aligned}$$

b) $\lim\limits_{x\to 1} \dfrac{(x^2-1)\sin(x-1)}{(x-1)^2}$

solution:

$$\lim_{x\to 1} \frac{(x^2-1)\sin(x-1)}{(x-1)^2} = \lim_{x\to 1} \frac{(x^2-1)}{x-1} \times \frac{\sin(x-1)}{(x-1)}$$
$$= \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} \times (1) = \lim_{x\to 1}(x+1) \times (1) = 2 \times 1 = 2$$

[**4**] **4.** Find each of the following derivatives. You may use any technique we have seen so far in the course. You do not need to simplify your answer.

a) $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{re^r + 2\pi^2}{r^3+1}\right)$

solution:

$$\frac{\left(re^r+2\pi^2\right)'\left(r^3+1\right) - \left(re^r+2\pi^2\right)\left(r^3+1\right)'}{\left(r^3+1\right)^2} = \frac{\left((1)(e^r)+r(e^r)+0\right)'\left(r^3+1\right) - \left(re^r+2\pi^2\right)\left(3r^2+0\right)}{\left(r^3+1\right)^2}$$

b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos(e^{(x^2)})\right)$

solution:

$$-\sin(e^{x^2}) \times e^{x^2} \times 2x$$

[**2**] **5.** Find the equation of all lines that are tangent to $f(x) = 2x^2$ and pass through the point $(0, -8)$.

(hint: find the slope of the tangent line to $f(x)$ at $x = a$, then find the equation of the tangent line at $x = a$, and then find the value(s) of a that make this line pass through the given point.)

solution: We find the derivative $f'(x) = 4x$. When $x = a$ the slope is $4a$. When $x = a$ we have $y = f(a) = 2a^2$. The line has slope $4a$ and passes through $(a, 2a^2)$. So the equation is given by:

$$\begin{aligned}(y-2a^2) &= 4a(x-a)\\ y &= 4ax - 2a^2\end{aligned}$$

The line is supposed to pass through $(0, -8)$, so we substitute this into the equation for the line (either version of the equation is fine).

$$\begin{aligned}(-8) &= 4a(0) - 2a^2\\ a^2 &= 4\\ a &= \pm 2\end{aligned}$$

So there are two lines.

$$y = 4(2)x - 2(2)^2 = 8x - 8$$
$$y = 4(-2)x - 2(-2)^2 = -8x - 8$$

Alternative solution:
Consider some point on the curve with $x = a$; then $y = 2a^2$ so we have the point $(a, 2a^2)$.

The slope at $x = a$ is $f'(a) = 4a$. The slope of the line connecting $(a, 2a^2)$ to $(0, -8)$ is $(2a^2 - (-8)) / (a - 0)$. The two slopes must be equal.

$$\begin{aligned} 4a &= \frac{2a^2 - (-8)}{a - 0} \\ 4a^2 &= 2a^2 + 8 \\ a^2 &= 4 \\ a &= \pm 2 \end{aligned}$$

We get the lines again

$$y - (-8) = 4a(x - 0) \begin{cases} y = 4ax - 8 = 8x - 8 \\ y = 4ax - 8 = -8x - 8 \end{cases}$$

[**2**] **6.** Find an expression for y' in terms of x and y, given that $y^4 = e^x + x^2$.

solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(y^4\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x + x^2\right) \\ 4y^3 \times y' &= e^x + 2x \\ y' &= \frac{e^x + 2x}{4y^3} \end{aligned}$$

[**2**] **7.** Consider the function $f(x) = x^{(e^x)}$. Find an expression for $f'(x)$ purely in terms of x. (hint: logarithmic differentiation.)

solution: using log-diff:

$$\begin{aligned} y &= x^{(e^x)} \\ \ln(y) &= \ln(x^{(e^x)}) = e^x \ln(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\ln(y)) &= \frac{\mathrm{d}}{\mathrm{d}x}(e^x \ln(x)) \\ \frac{1}{y} y' &= e^x \times 1/x + e^x \ln(x) \\ y' &= y\left(\frac{e^x}{x} + e^x \ln(x)\right) \\ y' &= x^{(e^x)}\left(\frac{e^x}{x} + e^x \ln(x)\right) \end{aligned}$$

or transforming and chain rule:

$$x^{(e^x)} = e^{\ln\left(x^{(e^x)}\right)} = e^{e^x \ln(x)}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{(e^x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^x \ln(x)}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{(e^x)}\right) = e^{e^x \ln(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x \ln(x)\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{(e^x)}\right) = e^{e^x \ln(x)}\left(e^x 1/x + e^x \ln(x)\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{(e^x)}\right) = x^{(e^x)}\left(\frac{e^x}{x} + e^x \ln(x)\right)$$

[3] **8.** A box with a square base is changing its height and the side of its base continuously. When the side of the base measures 2cm and the height 10cm, the side of the base is growing at the rate of 1cm/min while the height is decreasing at the rate of 2cm/min. What is the rate of change of the volume at that moment?

solution: Draw a picture of a box whose base measures a by a, and whose height measures h. The volume is $V = a \times a \times h = a^2h$. Differentiate:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(a^2h\right)$$
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\right)h + a^2\frac{\mathrm{d}}{\mathrm{d}t}(h)$$
$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2a\frac{\mathrm{d}a}{\mathrm{d}t}h + a^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

Everything on the RHS is a known quantity so we get $\frac{\mathrm{d}V}{\mathrm{d}t}$ directly. All units are in cm and sec, so we don't need to convert anything.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2(2)(1)(10) + (2)^2(-2) = 32$$