MAT 1004A **Term test #1** **1 October 2009**

Total marks: 30 points

Non-programmable calculators are permitted. This test is closed book.
Supply your answers on this sheet, but TA's have extra paper if you need it.

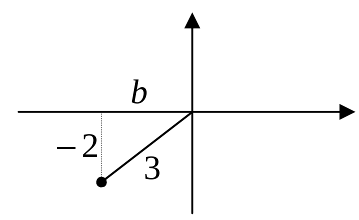
PLEASE PRINT **SOLUTION**
First name Last name Student number

Please show your work where appropriate!

1. [2 marks] Find the domain of ***h*** if: $h(x)=\dfrac{\sqrt{3-x}}{x^2+2x}=\dfrac{\sqrt{3-x}}{x(x+2)}$

1) $3-x\geq 0 \Leftrightarrow x\leq 3$
2) $x\neq 0$ $\therefore$ *dom f* : $x\in\{(-\infty,-2)\cup(-2,0)\cup(0,3]\}$
3) $x+2\neq 0 \Leftrightarrow x\neq -2$

2. Let θ be an angle in radians such that: $-\pi<\theta<-\pi/2$
   a. [1] The terminal side is in quadrant _3___
   b. [3] Find the **exact** ratios $\cos\theta$ and $\tan\theta$ if $\sin\theta=-2/3$



NB – **Show your work** (the diagram helps). By "**exact** ratios" it is meant a ratio of the form $\sqrt{3}/2$, etc… no decimals!

$b=-\sqrt{3^2-2^2}=-\sqrt{5}$

$\therefore \cos\theta=\dfrac{\sqrt{5}}{-3}=-\dfrac{\sqrt{5}}{3}$ *and* $\tan\theta=\dfrac{-2}{-\sqrt{5}}=\dfrac{2\sqrt{5}}{5}$

3. Let $f(x)=1/x^2$ and $g(x)=\sqrt{5-x}$ find a rule for:
   a. [2] $f\circ g(x)=f(g(x))$

$$f(g(x))=\frac{1}{(g(x))^2}=\frac{1}{\left(\sqrt{5-x}\right)^2}=\frac{1}{5-x}$$

   b. [2] $g\circ f\circ f(x)=g(f(f(x)))$

$$g\circ f\circ f(x)=g(f(f(x)))=\sqrt{5-(f(f(x)))}=\sqrt{5-\left(\frac{1}{[f(x)]^2}\right)}=\sqrt{5-\left(\frac{1}{[1/x^2]^2}\right)}=\sqrt{5-x^4}$$

4. Compute the following limits. If a limit is infinite, determine whether it is $+\infty$ or $-\infty$.
SHOW YOUR WORK.

   a. [2] $\lim\limits_{x\to\infty}\left(\dfrac{4x^2-3x+2}{7-10x-3x^2}\right)\left(\equiv\dfrac{\infty}{\infty}\right)\Rightarrow\begin{matrix}\div x^2\\ \div x^2\end{matrix}\Rightarrow\lim\limits_{x\to\infty}\left(\dfrac{4-3/x+2/x^2}{7/x^2-10/x-3}\right)=\left(\dfrac{4-0+0}{0-0-3}\right)=-\dfrac{4}{3}$

   b. [2]

$$\lim_{x\to-5}\left(\frac{x^2+x-20}{x^2+5x}\right)\left(\equiv\frac{0}{0}\right)\Rightarrow\lim_{x\to-5}\left(\frac{x^2+x-20}{x^2+5x}\right)=\lim_{x\to-5}\left(\frac{(x+5)(x-4)}{x(x+5)}\right)=\lim_{x\to-5}\left(\frac{(x-4)}{x}\right)=\frac{-5-4}{-5}=\frac{9}{5}$$

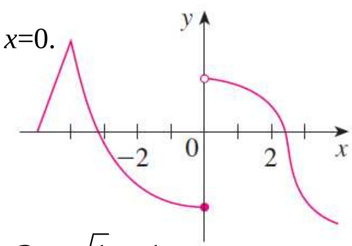
   c. [2]

$$\lim_{x\to-3^-}\left(\frac{x^2-x-20}{x+3}\right)\left(\equiv\frac{A}{0}\right)\Rightarrow\lim_{x\to-3^-}\left(\frac{x^2-x-20}{x+3}\right)=\lim_{x\to-3^-}\left(\frac{(x-5)(x+4)}{x+3}\right)=\frac{(-5^-)(+1^-)}{0^-}=\frac{(-)(+)}{(-)}=+\infty$$

5. Questions on continuity
   a. [2] Let f be a function. What conditions are necessary for f to be continuous at x=a?

   $$1)\ f(a)\ \textit{is defined} \qquad 2)\ \lim_{x\to a} f(x) = L\ (a\ \textit{finite no.}) \qquad 3)\ f(a) = \lim_{x\to a} f(x) = L$$

   b. [2] In the graph of g shown here, **state whether g is continuous or not** at x=0. Justify your answer using the necessary conditions in "a." as a basis.



   Conditions 2) and 3) are not met.

6. Let f be a function such that: $f(x) = \begin{cases} 2-\sqrt{x} & x \geq 1 \\ 2-(1/x) & 0 < x < 1 \end{cases} \Rightarrow f(1) = 2-\sqrt{1} = 1$

   a. [2] Show that f is continuous at x=1.

   $$\left.\begin{array}{l} \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(2-\sqrt{x}\right) = 2-\sqrt{1^+} = 1 \\ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(2-(1/x)\right) = 2-\left(1/1^-\right) = 1 \end{array}\right\} \Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 1 = f(1)$$

   Hence f is continuous at x=1

   b. [4] Show however, that f is NOT differentiable at x=1. Use the basic definition for $f_+'(a)$ and $f_-'(a)$ to show this. Remember: $f_+'(a) = \lim_{h\to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right)$.

   $$f_+'(1) = \lim_{h\to 0^+}\left(\frac{f(1+h)-f(1)}{h}\right) = \lim_{h\to 0^+}\left(\frac{2-\sqrt{1+h}-1}{h}\right) = \lim_{h\to 0^+}\left(\frac{1-\sqrt{1+h}}{h}\right) =$$

   $$\cdots = \lim_{h\to 0^+}\left(\frac{1-\sqrt{1+h}}{h} \bullet \frac{\left(1+\sqrt{1+h}\right)}{\left(1+\sqrt{1+h}\right)}\right) = \lim_{h\to 0^+}\left(\frac{1-(1+h)}{h\left(1+\sqrt{1+h}\right)}\right) = \lim_{h\to 0^+}\left(\frac{-h}{h\left(1+\sqrt{1+h}\right)}\right) =$$

   $$\cdots = \lim_{h\to 0^+}\left(\frac{-1}{\left(1+\sqrt{1+h}\right)}\right) = -\frac{1}{1+\sqrt{1}} = -\frac{1}{2}$$

   $$f_-'(1) = \lim_{h\to 0^-}\left(\frac{f(1+h)-f(1)}{h}\right) = \lim_{h\to 0^-}\left(\frac{2-\frac{1}{1+h}-1}{h}\right) = \lim_{h\to 0^-}\left(\frac{1-\frac{1}{1+h}}{h}\right) = \lim_{h\to 0^-}\left(\frac{\frac{1+h-1}{(1+h)}}{h}\right) =$$

   $$\cdots = \lim_{h\to 0^-}\left(\frac{h}{h(1+h)}\right) = \lim_{h\to 0^-}\left(\frac{1}{(1+h)}\right) = \frac{1}{1+0} = 1$$

   $\therefore f_-'(1) \neq f_+'(1)$ ... thus f is not differentiable at x=1

7. **Modify the left side** to prove the following trigonometric identities:

   a. [2] $\sec x - \cos x = \tan x \sin x$

   $$\sec x - \cos x = \frac{1}{\cos x} - \cos x = \frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \frac{\sin x}{\cos x} \bullet \sin x = \tan x \sin x$$ **QED**

   b. [2] $\frac{2\tan x}{1+\tan^2 x} = \sin 2x \Rightarrow \frac{2\tan x}{1+\tan^2 x} = \frac{2\sin x}{\cos x(1+\tan^2 x)} = \frac{2\sin x}{\cos x(\sec^2 x)} = 2\sin x\cos x = \sin 2x$

   QED