Math 1002 Assignment 2
Solutions.

1. c) $\sup\{2,7\} = 7$
   $\inf\{2,7\} = 2$

   b) $\sup(0,1) = 1$
   $\inf(0,1) = 0$

   e) $\sup\{\frac{1}{n} : n \in \mathbb{N}\} = 1$
   $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$

   f) $\sup\{0\} = 0$
   $\inf\{0\} = 0$

   j) $\sup\{1-\frac{1}{3^n} : n \in \mathbb{N}\} = 1$
   $\inf\{1-\frac{1}{3^n} : n \in \mathbb{N}\} = 2/3$

   k) $\sup\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\} = \infty$ (or sup DNE)
   $\inf\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\} = \inf\{0, 5/2, 8/3, \dots\} = 0$

   m) $\sup\{r \in \mathbb{Q} : r^2 < 4\} = 2$
   $\inf\{r \in \mathbb{Q} : r^2 < 4\} = -2$

   t) $\sup\{x \in \mathbb{R} : x^3 < 8\} = 2$
   $\inf\{x \in \mathbb{R} : x^3 < 8\} = -\infty$ (or inf DNE)

2. a) Suppose S is a nonempty bounded subset of $\mathbb{R}$. Then since S is bounded above and below, $\inf S$ and $\sup S$ are real numbers.
   If $s \in S$, then $\inf S \le s$ and $\sup S \ge s$. Hence
   $\inf S \le s \le \sup S \Rightarrow \inf S \le \sup S$

   b) If $\inf S = \sup S = M$, then we must have $S = \{M\}$.

<u>Proof:</u> We prove that $S \subseteq \{M\}$ and $\{M\} \subseteq S$.

$S \subseteq \{M\}$: Suppose $s \in S$. Then there are three possibilities.

$s < M$: In this case M is not a lower bound for S, a contradiction

$s > M$: In this case M is not an upper bound for S, a contradiction

The only other possibility is $s = M \Rightarrow S \subseteq \{M\}$

Since S is assumed to be nonempty, this actually proves $S = \{M\}$, because $\{M\}$ is the only nonempty subset of $\{M\}$. ∎

3. Let $a \in \mathbb{R}$. We show $\sup(-\infty, a) = a$.

(i) First we show a is an upper bound.

This is by definition of $(-\infty, a)$, as $x \in (-\infty, a) \Leftrightarrow x < a$

(ii)' Now, suppose $M < a$. We need to find some element of $(-\infty, a)$ which is larger than M.

Since $M < a$, the open interval (M, a) contains a rational number $r \in (M, a)$ by denseness of $\mathbb{Q}$. Then $M < r < a$. Since $r < a$, $r \in (-\infty, a)$. Since $r > M$, we have satisfied (ii)'.

Since (i) and (ii)' are both satisfied, $\sup(-\infty, a) = a$.

4. This is straight forward using the (ii) version of the definition I told you to formulate. Since I didn't state it, here it is:

$m = \inf S$ if two things hold

(i) m is a lower bound for S, and

(ii) If m' is a lower bound for S, then $m' \leq m$.

This should be justified from the textbook def:

$m = \inf S$ if two things hold

(i) m is a lower bound for S, and

(ii)' If $x > m$, then there exists $s \in S$ with $s < x$.

Pf of $(i) + (ii)' \Rightarrow (i) + (ii)$

Suppose that (i) and (ii) hold, and suppose m' is a lower bound for S. Aiming to get a contradiction, suppose $m' > m$. Then (ii) $\Rightarrow$ we can find $s \in S$ with $s < m'$, which contradicts m' being a lower bound. Hence $m' \leq m$.

With that proven, we can prove $\inf S$ is unique.

Suppose M and N are both greatest lower bounds of S.

(ii) $\Rightarrow$ $M \leq N$, since N is the greatest lower bound

(ii) $\Rightarrow$ $N \leq M$, since M is the greatest lower bound

$\Rightarrow$ $M = N$ ▨

Alternate proof: Can prove this just using (ii)', of course.

Suppose $\inf S = M$ and $\inf S = N$

Suppose $M < N$. Then since M is the inf, we can find $x \in S$ with $x < N$. This contradicts N being a lower bound, so $M \geq N$.

Suppose $N < M$. Then since N is the inf, we can find $x \in S$ with $x < M$. This contradicts M being a lower bound, so $N \leq M$.

$\Rightarrow$ $M = N$

Suggested exercises

(i) Since $-3<x<4$ for all $x \in (-3,4)$, 4 is an upperbound.

Now, suppose $M<4$. There are two cases

Case 1: $M \leq -3$

In this case, $x=0$ is in $(-3,4)$ and satisfies $M<x$.

Case 2: $M>-3$

In this case, denseness of $\mathbb{Q}$ implies existence of a rational number r between M and 4, $M<r<4$.

Since $-3<M<r<4$, we have $r \in (-3,4)$ and $M<r$

In both cases, we are able to find $x \in (-3,4)$ with $M<x$.

Hence $\sup(-3,4)=4$

(ii) The proof works exactly the same as in Q2 on Tutorial 2. The difference between $[a,\infty)$ and (a,∞) doesn't change the proof at all.

(iii) Let $a \in \mathbb{R}$, $b \in \mathbb{R}$, with $a<b$. Prove there are infinitely many rationals between a and b.

Pf: Suppose that the result is false, and that there are only finitely many elements in $\{r \in \mathbb{Q} \mid a<r<b\} := A$

Since A is finite, it has a minimum. Let $R = \min A$.

Then R is rational, and $a<R<b$. Since $a<R$, denseness of $\mathbb{Q}$ implies existence of a rational r such that $a<r<R$. But then this implies $r \in A$, and $R>r$, contradicting that R was the min. Hence A must be infinite.

(iv) a) $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{16}$

b) $\frac{4}{3}, \frac{7}{7}, \frac{10}{11}, \frac{13}{15}, \frac{16}{19}$

c) $\frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \frac{5}{243}$

d) $\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}$