Assignment 4

1. Let $M<0$. We need $-3s_n < M \iff s_n > \frac{-M}{3}$ (since $-3<0$)

Since $-\frac{M}{3} > 0$ and $\lim_{n\to\infty} s_n = \infty$, we know there exists N such that $n>N$ implies $s_n > -\frac{M}{3}$.

For all such n we have $s_n > -\frac{M}{3} \Rightarrow -3s_n < M$,

as required. ▨

2. a) Rewrite

$$s_n = \frac{a_k + a_{k-1}\cdot\frac{1}{n} + \cdots + a_0\frac{1}{n^k}}{b_k + b_{k-1}\cdot\frac{1}{n} + \cdots + b_0\frac{1}{n^k}}$$

Now

$$\lim_{n\to\infty} a_k + a_{k-1}\cdot\frac{1}{n} + \cdots + a_0\frac{1}{n^k} = \lim_{n\to\infty} a_k + \lim_{n\to\infty} a_{k-1}\frac{1}{n} + \cdots + \lim_{n\to\infty} a_0\frac{1}{n^k} \quad \text{By Thm 9.3}$$

$$= a_k + a_{k-1}\lim_{n\to\infty}\frac{1}{n} + \cdots + a_0\lim_{n\to\infty}\frac{1}{n^k} \quad \text{By Thm 9.2, 9.1}\tfrac{1}{2}$$

$$= a_k + 0 + 0 + \cdots + 0 \quad \text{By Thm 9.7}$$

$$= a_k$$

Similarly, $\lim_{n\to\infty} b_k + \cdots + b_0\frac{1}{n^k} = b_k$ (by the same Theorems)

Hence Thm 9.6 $\Rightarrow \lim_{n\to\infty} s_n = \frac{a_k}{b_k}$

b) As above, rewrite (multiplying top and bottom by $\frac{1}{n^\ell}$)

$$S_n = \frac{a_k \frac{1}{n^{\ell-k}} + a_{k-1} \frac{1}{n^{\ell+1-k}} + \cdots + a_0 \frac{1}{n^\ell}}{b_k + b_{k-1} \frac{1}{n} + \cdots + b_0 \frac{1}{n^k}}$$

Thm 9.2, 9.3

But

$$\lim_{n\to\infty} a_k \frac{1}{n^{\ell-k}} + a_{k-1} \frac{1}{n^{\ell+1-k}} + \cdots + a_0 \frac{1}{n^\ell} = a_k \lim_{n\to\infty} \frac{1}{n^{\ell-k}} + a_{k-1} \lim_{n\to\infty} \frac{1}{n^{\ell+1-k}} + \cdots + a_0 \lim_{n\to\infty} \frac{1}{n^\ell}$$

$$= a_k \cdot 0 + a_{k-1} \cdot 0 + \cdots + a_0 \cdot 0 \quad \text{Thm 9.7}$$

$$= 0$$

As in (a), $\lim_{n\to\infty} b_\ell + b_{\ell-1} \frac{1}{n} + \cdots + b_0 \frac{1}{n^\ell} = b_\ell \neq 0$

Hence $\lim_{n\to\infty} S_n = \frac{0}{b_\ell}$ Thm 9.6

$= 0$.

c) We are assuming $\frac{a_k}{b_k} > 0$, so a_k and b_k have the same sign. If they are both negative, multiply top and bottom by -1 and rename to get $a_k > 0$ and $b_k > 0$.

We factor out the extra powers of n from the top

$$S_n = \frac{n^{k-\ell}\left(a_k n^\ell + a_{k-1} n^{\ell-1} + \cdots + a_0 \frac{1}{n^{k-\ell}}\right)}{b_\ell n^\ell + b_{\ell-1} n^{\ell-1} + \cdots + b_0}$$

$$= n^{k-\ell} \left(\frac{a_k + a_{k-1} \frac{1}{n} + \cdots + a_0 \frac{1}{n^k}}{b_\ell + b_{\ell-1} \frac{1}{n} + \cdots + b_0 \frac{1}{n^\ell}} \right)$$

$$\lim_{n\to\infty} a_k + a_{k-1}\frac{1}{n} + \dots + a_0 \frac{1}{n^k} = a_k$$

$$\lim_{n\to\infty} b_\ell + b_{\ell-1}\frac{1}{n} + \dots + b_0 \frac{1}{n^\ell} = b_\ell$$

By Thm 9.1$\frac{1}{2}$, 9.2, 9.3, 9.7 as before

$$\Rightarrow \lim_{n\to\infty}\left(\frac{a_k + a_{k-1}\frac{1}{n} + \dots + a_0\frac{1}{n^k}}{b_\ell + b_{\ell-1}\frac{1}{n} + \dots + b_0\frac{1}{n^\ell}}\right) = \frac{a_k}{b_\ell} > 0 ! \text{ by Thm 9.6.}$$

Since $k-\ell > 0$, we know $n^{k-\ell} \geq n$ for all $n \in \mathbb{N}$,
so if $M > 0$, choosing $N = M$ means

$$n > N \Rightarrow n^{k-\ell} \geq n > N = M \Rightarrow \lim_{n\to\infty} n^{k-\ell} = \infty.$$

Hence Thm 9.9 implies

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (n^{k-\ell})\left(\frac{a_k + \dots + a_0\frac{1}{n^k}}{b_\ell + \dots + b_0\frac{1}{n^\ell}}\right) = +\infty.$$

3. For all $n \geq 1$ we have

$$\begin{aligned} 3n - 4n^3 &\leq 3n^3 - 4n^3 && \text{since } n^3 \geq n \\ &= -n^3 \\ &\leq -n && \text{since } n^3 \geq n \Rightarrow -n^3 \leq -n \end{aligned}$$

So let $M < 0$ and take $N = -M$. Then $n > N$ implies

$$\begin{aligned} n > -M \Rightarrow M &> -n \\ &\geq 3n - 4n^3 && \text{as above.} \end{aligned}$$

Since M was arbitrary, $\lim_{n\to\infty} 3n - 4n^3 = -\infty$.