MCG 4308 _ MIDTERM EXAM

Problem 1

The fact that the sistem is on an inclined plane is irrelevant with respect to the questions.

(1) Energy

Kinetic energy : $T = \frac{1}{2} M \dot{x}_2^2$

Potential energy : $U = \frac{1}{2} k_{eq} x_2^2$

$K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$ (series)

It can be derived by assuming same force

$F_{K_1} = F_{K_2}$

$F_{K_{eq}} = F_{K_1} = F_{K_2}$

$K_1 x_1 = K_2 (x_2 - x_1) \Rightarrow (K_1 + K_2) x_1 = K_2 x_2$

$K_{eq} x_2 = K_1 x_1 \Rightarrow K_{eq} x_2 = \frac{K_1 K_2}{K_1 + K_2} x_2$

$\Rightarrow K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$

Therefore

$$T = \frac{1}{2} M \dot{x}_2^2$$

$$U = \frac{1}{2} \frac{K_1 K_2}{K_1 + K_2} x_2^2$$

Since this is an harmonic oscillator

$$x_2(t) = A \sin(\omega_n t + \phi)$$

Using $U_{MAX} = T_{MAX}$

$$T_{MAX} = \frac{1}{2} M \omega_n^2 A^2$$

$$U_{MAX} = \frac{1}{2} \frac{K_1 K_2}{K_1 + K_2} A^2$$

$$\Rightarrow \quad \omega_n = \sqrt{\frac{K_1 K_2}{M(K_1 + K_2)}}$$

Mathematical model

$$\frac{d}{dt}(\mathcal{U}+T)=0 \implies \dot{x}_2\left(M\ddot{x}_2+\frac{k_1k_2}{k_1+k_2}x_2\right)=0$$

$$\implies M\ddot{x}_2+\frac{k_1k_2}{k_1+k_2}x_2=0$$

Annex: does gravity matter?

Let Δ be de Mass' deflection due to gravity.

At the equilibrium

$$F_k - Mg\sin\beta = 0$$

$$N - Mg\cos\beta = 0$$

$$F_{k_{eq}} = -K_{eq}\Delta$$

$$Mg\sin\beta = -k_{eq}\Delta$$

Potential energy with static eq. 4

$$\mathcal{U} = \frac{1}{2} k_{eq} (x_2 + \Delta)^2 + mg \cos\beta (x_2 + \Delta)$$

$$T = \frac{1}{2} M \left[\frac{d}{dt}(x_2 + \Delta)\right]^2 = \frac{1}{2} M \dot{x}_2^2$$

Equation of motion

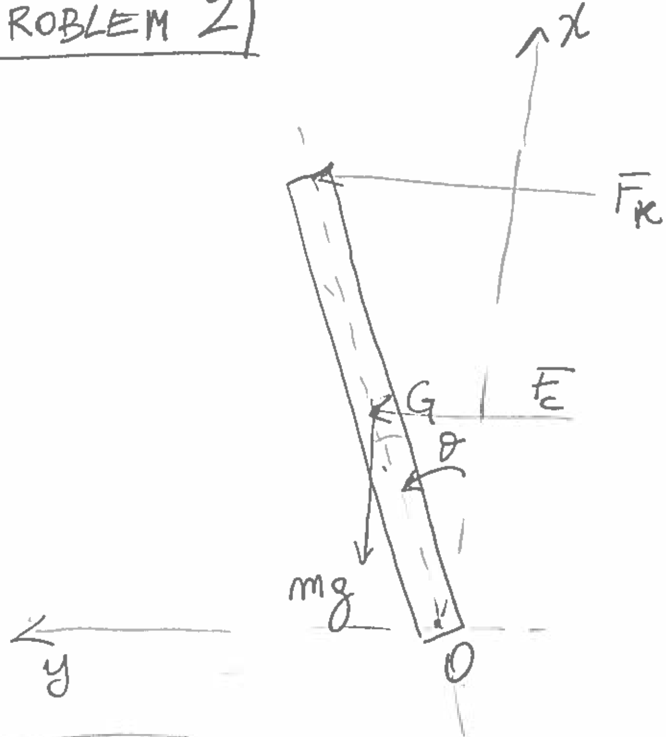
$$\frac{d}{dt}(\mathcal{U} + T) = 0 \Rightarrow \dot{x}_2 M \ddot{x}_2 + K_{eq}(x_2 + \Delta)\dot{x}_2 + \dot{x}_2 \, mg \cos\beta = 0$$

$$\Rightarrow \dot{x}_2 (M \ddot{x}_2 + K_{eq} x_2) + \underbrace{(mg\cos\beta + K_{eq}\Delta)}_{\substack{=0 \\ \text{static equilibr.}}} = 0$$

Therefore the natural frequency is

$$\omega_n = \sqrt{\frac{K_{eq}}{M}} = \sqrt{\frac{K_1 K_2}{M(K_1 + K_2)}}$$

PROBLEM 2



Point 1

Moment around point O

$$J_0 \ddot{\theta} = F_c L \cos\theta + 2 L F_k \cos\theta + mg L \sin\theta$$

$$J_0 = J_G + L^2 m$$

$$J_G = \frac{m}{12}(b^2 + 4L^2) = \frac{mL^2}{3}\left(\frac{b^2}{4L^2} + 1\right)$$

The reaction forces are

$$F_c = -c\dot{y}_G = -c\dot{\theta}L\cos\theta$$

$$F_k = -k y_L = -2kL\sin\theta$$

Therefore

$$J_0 \ddot{\theta} + cL^2 \dot{\theta} \cos^2\theta + \left(4L^2 K \cos\theta - mgL\right) \sin\theta$$

For small θ

$$J_0 \ddot{\theta} + cL^2 \dot{\theta} + \left(4L^2 K - mgL\right) \theta = 0$$

Dividing by L^2

$$m\left(\frac{4}{3} + \frac{b^2}{12L^2}\right)\ddot{\theta} + c\dot{\theta} + \left(4K - \frac{mg}{L}\right)\theta = 0$$

In order for the solution to be asymptotically stable all coeff. must be positive. Therefore

$$K > \frac{mg}{4L}, \quad m > 0, \quad c > 0$$

Problem 3

At the peak, the frequency ratio is

$$r_{peak} = \sqrt{1-2\zeta^2}$$

From the plot, $r_{peak} \sim 0.9$. Therefore

$$\zeta^2 = \frac{1-r_{peak}^2}{2} = \frac{0.1}{2} = \frac{1}{20} = 0.05$$

$$\zeta = \sqrt{0.05} = 0.23$$