STUDENT NAME: **RUBRIC**

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work—your work must justify your answer, and clearly identify your final answer. No books, notes, or electronic devices of any kind may be used during the exam period. You must simplify results of function evaluations when it is reasonable to do so. For example,*** $4^{3/2}$ ***should be evaluated (replaced by*** 8***).***

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1. Calculate the following limits, or explain why they do not exist.

   (a) [3 pts] $\displaystyle\lim_{x\to -2} \frac{\sqrt{x+4}}{\cos(\pi x)}$.

   This function is continuous at $x = -2$, so we just plug in:

   $$= \frac{\sqrt{-2+4}}{\cos(-2\pi)} = \frac{\sqrt{2}}{1}.$$

   (b) [3 pts] $\displaystyle\lim_{t\to 2} \frac{t^2+2t-8}{t^2-4}$.

   Plugging in gives $0/0$, which tells us that there is a common factor to cancel. Factoring yields $\displaystyle\lim_{t\to 2} \frac{(t-2)(t+4)}{(t-2)(t+2)} = \lim_{t\to 2} \frac{(t+4)}{t+2} = 3/2.$

   (c) [3 pts] $\displaystyle\lim_{x\to \infty} \frac{\pi x - 1000x^2 + 8x^3}{-3x^3 + 1 - 4x}$.

   We compare the largest powers in the numerator and denominator, which are both 3. Hence, the limit at infinity is the ratio of the coefficients of these powers, giving $-8/3$.

(d) [4 pts] $\lim_{x\to 2} \frac{|x-2|}{4x-8}$

Since the function involves an absolute value which changes definition at $x = 2$, we analyze one-sided limits. From the left, we have $|x-2| = -(x-2)$, so

$$\lim_{x\to 2^-} \frac{|x-2|}{4x-8} = \lim_{x\to 2} \frac{-(x-2)}{4(x-2)} = -\frac{1}{4}.$$

From the right, we have $|x-2| = x-2$, and the same calculation gives

$$\lim_{x\to 2^+} \frac{|x-2|}{4x-8} = \frac{1}{4}.$$

Since the right and left limits do not agreee, this limit DNE.

2. [4 pts] Let $f(x) = \begin{cases} \frac{1}{4-x}, & x < 3 \\ ax+7, & x \geq 3 \end{cases}$ Find the value of a such that $\lim_{x\to 3} f(x)$ exists. Justify your answer.

We check that the one-sided limits agree in order to see that the limit exists. From the left, we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} \frac{1}{4-x} = 1.$$

From the right, we have

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} ax + 7 = 3a + 7.$$

So, we must have $3a + 7 = 1$, which implies that $a = -2$.

3. Let $h(x) = x^2 f(x) - 2g(2x)$, where $f(x)$ and $g(x)$ satisfy

$$\begin{array}{lll} f(-1) = 2 & g(-1) = 1 & g(-2) = 3 \\ f'(-1) = \pi & g'(-1) = 0 & g'(-2) = \frac{1}{2} \end{array}$$

(a) [4 pts] Find $h'(-1)$.

We first differentiate $h(x)$:

$$h'(x) = 2xf(x) + x^2 f'(x) - 4g'(2x).$$

Here we have used the product rule on the first term and the chain rule on the second. Now we plug in $x = -1$:

$$h'(-1) = -2f(-1) + (-1)^2 f'(-1) - 4g'(-2) = \pi - 6.$$

(b) [3 pts] Find an equation of the tangent line to $h(x)$ at $x = -1$. If you could not answer part (a), assume that the solution to part (a) is 5 for the purpose of solving this problem.

We know that the slope of the tangent line to $h(x)$ at $x = -1$ is $\pi - 6$ from part (a) (or 5, if you used the alternative assumption). We need to find a point on the tangent line. This point is $(-1, h(-1)) = (-1, f(-1) - 2g(-2)) = (-1, -4)$. Hence the equation of the tangent line is $y + 4 = (\pi - 6)(x + 1)$. If you used the assumption, it would be $y + 4 = 5(x + 1)$.

4. Find the derivatives of the following functions. Do not simplify your answers.

   (a) [3 pts] $f(x) = 5x^3 - \dfrac{4}{x} + 3\sqrt[3]{x}$.

   Using the power rule, we have $f'(x) = 15x^2 + \dfrac{4}{x^2} + x^{-2/3}$.

   (b) [3 pts] $g(x) = \dfrac{4x^3 + 2x - 1}{\sqrt{2x+1}}$.

   Using the quotient rule and the chain rule on the denominator, we have

$$g'(x) = \frac{(12x^2 + 2)(\sqrt{2x+1}) - (4x^3 + 2x - 1)(2 \cdot \frac{1}{2} \cdot (2x+1)^{-1/2})}{2x+1}.$$

5. (a) [3 pts] Let $f(x) = \frac{x^3}{3} - x^2 + x$. Find all values of x at which $f(x)$ has a horizontal tangent line, or explain why no such values exist.

The slope of a horizontal line is 0. The slope of the tangent line at $x = a$ is given by $f'(a)$. We have

$$f'(x) = x^2 - 2x + 1.$$

Setting this equal to 0 yields $(x-1)(x-1) = 0$, so $x = 1$ is the only value of x at which the tangent line is horizontal.

(b) [3 pts] Let $g(x) = \pi^2 - 3\pi + 2$. Find all values of x at which the tangent line to $g(x)$ has slope 1/2, or explain why no such values exist.

We have $g'(x) = 0$, as g is a constant function. Hence there are no values of x at which $g'(x) = 1/2$.

6. [2 pts] Find the slope of the secant line to $h(t) = \frac{t}{t+1}$ on the interval $[1, 3]$.

The slope of the secant line to a function on an interval is the average rate of change of the function on that interval. We have

$$\frac{h(3) - h(1)}{3 - 1} = \frac{3/4 - 1/2}{2} = \frac{1}{8}.$$

7. (a) [4 pts] Use *the limit definition of the derivative* to calculate the derivative of $g(x) = \dfrac{1}{1-x}$.

We have

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1-(x+h)} - \frac{1}{1-x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{1-x-1+(x+h)}{(1-x)(1-(x+h))}}{h} \\ &= \lim_{h\to 0} \frac{1}{(1-x)(1-(x+h))} = \frac{1}{(1-x)^2}. \end{aligned}$$

(b) [2 pts] Find an equation of the tangent line to $g(x)$ at $x = 3$. If you could not answer part (a), assume that $g'(x) = \dfrac{\pi}{x}$ for the purpose of solving this problem.

The slope of the tangent line at $x = 3$ *is* $g'(3) = \frac{1}{(1-3)^2} = \frac{1}{4}$. *A point on this line is* $(3, g(3)) = (3, -\frac{1}{2})$. *Hence the equation is* $y + 1/2 = \frac{1}{4}(x-3)$. *If you used the alternative assumption, the slope is* $\frac{\pi}{3}$, *and the equation is* $y + 1/2 = \frac{\pi}{3}(x-3)$.

8. During the construction of an office building, a hammer is accidentally dropped from a height of 90 feet. The distance (in feet) the hammer falls in t seconds is $s(t) = 10t^2$. (This is NOT the height of the hammer off of the ground, but the distance that it has fallen.)

   (a) [2 pts] What is the hammers velocity when it strikes the ground?

   The hammer hits the ground when it has traveled 90 feet, or when $s(t) = 90$. *This happens at* $t = 3$. *The velocity function is* $v(t) = s'(t) = 20t$, *so the impact velocity is* $v(3) = 60$ *ft/s.*

   (b) [2 pts] What is the hammer's acceleration when it hits the ground?

   The acceleration function is $a(t) = v'(t) = 20$, *so the acceleration at all times is 20 ft/s*^{2}.

   (c) [2 pts] What is the average velocity of the hammer throughout the duration of its fall?

The fall starts at $t = 0$ *and ends at* $t = 3$. *Thus the average velocity (which is the average rate of change of position) is*

$$\frac{s(3) - s(0)}{3 - 0} = \frac{90 - 0}{3} = 30 ft/s$$