MAT 1320A– Test 1 9 October 2019

Professor: Ali Karimnezhad

First and Last Name ________________________________

Student Number ________________________________

Circle your DGD:

DGD1 (08:30-09:50 MRT 251)
DGD2 (10:00-11:20 MRT 221)
DGD3 (11:30-12:50 CRX C309)
DGD4 (13:00-14:20 MRT 251)
DGD5 (14:30-15:50 MRT 219)

Instructions:

- Print your name and student number on this page.
- The duration of the exam is 80 minutes.
- No calculators are permitted. No notes, books, papers of any kind, or any other aids. It is forbidden to use or have in your possession a cellular telephone or other electronic device. Turn off your devices and put them in your bag.
- Make sure you have 8 questions. Do not detach the questionnaire.
- Questions 1 to 6 are multiple choice questions. These questions are worth 2 points each, and partial points are not awarded. Please write your answers in the corresponding box of the grid below.
- Questions 7 and 8 require a complete solution, so arrange your time accordingly. The correct answer requires a written justification that is legible and logical. You must show all of your work.
- You can use the back side of the pages as rough, or answer sheets clearly indicating it.

List your answers to multiple-choice questions:

Q1	Q2	Q3	Q4	Q5	Q6

DCBFDB

Do not write anything in the following table:

MCQ	Q7	Q8	Total
/12	/4	/4	/20

Part 1: Multiple-choice Questions

1. (2 points) Assume some values of a one-to-one function $y = f(x)$ are given in the following table:

x	1	2	3	4	5
$f(x)$	4	5	1	2	3
$g(x)$	2	4	5	3	1

Which one of the following statements is true?

A. $(f \circ g)(2) = 1,\ f^{-1}(2) = 1$
B. $(f \circ g)(2) = 2,\ f^{-1}(2) = 5$
C. $(f \circ g)(2) = 1,\ f^{-1}(2) = 4$
D. $(f \circ g)(2) = 2,\ f^{-1}(2) = 4$
E. $(f \circ g)(2) = 1,\ f^{-1}(2) = 5$
F. $(f \circ g)(2) = 2,\ f^{-1}(2) = 1$

Solution. (D) $(f \circ g)(2) = f(g(2)) = f(4) = 2$. Since $f(4) = 2$, $f^{-1}(2) = 4$.

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2. (2 points) Consider the following function

$$f(x) = \begin{cases} ax^2 + 2x, & \text{if } x \leq 2; \\ x^3 - ax, & \text{if } x > 2. \end{cases}$$

For what value of the constant a is the function f continuous on $(-\infty, \infty)$?

A. $a = 1$ B. $a = \frac{1}{2}$ C. $a = \frac{2}{3}$ D. $a = \frac{3}{2}$ E. $a = 2$
F. $a = 4$

Solution. (C) This function is continuous when $x < 2$ and when $x > 2$. When $x = 2$, $\lim_{x \to 2^-} ax^2 + 2x = 4a + 4$ and $\lim_{x \to 2^+} x^3 - ax = 8 - 2a$. Hence, $4a + 4 = 8 - 2a$ which results in $6a = 4$ and thus, $a = \frac{2}{3}$.

3. (2 points) If $2^x = 3^{x+1}$, then $x =$

A. $\dfrac{\ln 3}{\ln(3/2)}$ B. $\dfrac{\ln 3}{\ln(2/3)}$ C. $\dfrac{\ln 2}{\ln(3/2)}$ D. $\dfrac{\ln 2}{\ln(2/3)}$ E. $\dfrac{3}{\ln(3/2)}$

F. $\dfrac{\ln(3/2)}{\ln 3}$

Solution. (B) Take the natural logarithm on both sides of the equation: $\ln(2^x) = \ln(3^{x+1})$. By the property of the logarithm, $x \ln 2 = (x+1)\ln 3$. Thus, $x(\ln 2 - \ln 3) = \ln 3$ and hence, $x = \frac{\ln 3}{(\ln 2 - \ln 3)} = \frac{\ln 3}{(\ln \frac{2}{3})}$.

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4. (2 points) Find the one-sided limit: $\displaystyle\lim_{x \to -1^-} \frac{-3x^2 - x + 2}{|x+1|} =$

A. 1 B. -1 C. 3 D. -3 E. 5 F. -5

Solution. (F)

$$\begin{aligned}
\lim_{x \to -1^-} \frac{-3x^2 - x + 2}{|x+1|} &= \lim_{x \to -1^-} \frac{-3x^2 - x + 2}{-(x+1)} \\
&= \lim_{x \to -1^-} \frac{(-3x+2)(x+1)}{-(x+1)} \\
&= \lim_{x \to -1^-} -(-3x+2) = -5
\end{aligned}$$

5. (2 points) If $f(x) = e^{-2x}\cos\left(\dfrac{\pi}{2}x\right)$, then the equation of the tangent line of the graph of $f(x)$ at the point $(1, e^{-2})$ is

A. $y = -e^{-2}x + \frac{\pi}{2}e^{-2}$

B. $y = -\frac{\pi}{2}e^{-2}x + \frac{\pi}{2}e^{-2}$

C. $y = e^{-2}x + \frac{\pi}{2}e^{-2} + 1$

D. $y = -\frac{\pi}{2}e^{-2}x + \frac{\pi+2}{2}e^{-2}$

E. $y = \frac{\pi}{2}e^{-2}x + \frac{3\pi}{2}e^{-2}$

F. $y = e^{-2}x + \frac{\pi}{2}e^{-2}$

Solution. (D) By the chain rule, $\dfrac{d}{dx}e^{-2x} = -2e^{-2x}$, and $\dfrac{d}{dx}\cos\left(\frac{\pi}{2}x\right) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)$. Hence, by the product rule, $f'(x) = -2e^{-2x}\cos\left(\frac{\pi}{2}x\right) + e^{-2x}\left(-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)\right)$. Now, when $x = 1$, $f'(1) = -2e^{-2}\cos\left(\frac{\pi}{2}\right) + e^{-2}\left(-\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)\right) = -\frac{\pi}{2}e^{-2}$. The equation of the tangent line of the graph of $f(x)$ at the point $(1, e^{-2})$ is $y - e^{-2} = -\frac{\pi}{2}e^{-2}(x-1)$, or $y = -\frac{\pi}{2}e^{-2}x + (\frac{\pi}{2}+1)e^{-2} = -\frac{\pi}{2}e^{-2}x + (\frac{\pi+2}{2})e^{-2}$.

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6. (2 points) Suppose a function $y = f(x)$ is defined implicitly by the equation

$$\frac{x}{y} - 2x + 9y^2 = 3$$

near a point $(2, -1)$. Then the derivative of this function at the point $(2, -1)$ is

A. $\dfrac{3}{20}$ B. $-\dfrac{3}{20}$ C. $\dfrac{21}{2}$ D. $-\dfrac{21}{2}$ E. $\dfrac{1}{2}$ F. $-\dfrac{1}{2}$

Solution. (B) Taking the derivative on both sides with respect to x, we have $\dfrac{y - xy'}{y^2} - 2 + 18yy' = 0$, which by substituting $x = 2$ and $y = -1$, it reduces to $\dfrac{-1-2y'}{(-1)^2} - 2 + 18(-1)y' = 0$ or $y' = -\dfrac{3}{20}$.

Part II. Detailed-Answer Questions

7. (4 points) Consider the function $f(x) = \dfrac{x^2+x}{3x^2-2x-5}$

   (a) (2 points) Find the horizontal/vertical asymptote(s) of this graph, if any. (Answer NONE if there is no horizontal or vertical asymptote.)
   Answer. The horizontal asymptote(s) of the graph of $f(x)$ is / are

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   (b) (2 points) The vertical asymptote(s) of the graph of $f(x)$ is / are

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   Justification.
   (Show why what you give are vertical/horizontal asymptotes, and there is no other asymptote.)

Solution. Part (a)

$$\lim_{x\to\infty} \frac{x^2+x}{3x^2-2x-5} = \lim_{x\to\infty} \frac{1+\dfrac{1}{x}}{3-2\dfrac{1}{x}-5\dfrac{1}{x^2}} = \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{x^2+x}{3x^2-2x-5} = \lim_{x\to-\infty} \frac{1+\dfrac{1}{x}}{3-2\dfrac{1}{x}-5\dfrac{1}{x^2}} = \frac{1}{3}$$

Thus, the graph of $f(x)$ has one horizontal asymptote $y = \dfrac{1}{3}$.

Part (b) Let $3x^2-2x-5=0$. This gives $x=-1$ and $x=\dfrac{5}{3}$. Now, we need to verify if the limit of $f(x)$ is infinity when x approaches $x=-1$ and $x=\dfrac{5}{3}$. Since $\lim_{x\to-1} \dfrac{x^2+x}{3x^2-2x-5} = \lim_{x\to-1} \dfrac{x(x+1)}{(3x-5)(x+1)} = \dfrac{1}{8}$, $x=-1$ is not a vertical asymptote. Thus, the graph of $f(x)$ has only one vertical asymptote $x=\dfrac{5}{3}$.

8. (a) (1 point) Give the definition of the derivative of a function $y = f(x)$ at a point $x = a$.

(b) (3 points) Use the definition of the derivative to find the derivative of the function $y = f(x) = \sqrt{5x+4}$ at $x = 1$.
(Pay attention to your presentation. It counts!)

Solution. Part (a)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Part (b)

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{5(a+h)+4} - \sqrt{5a+4}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5(a+h)+4} - \sqrt{5a+4}}{h} \times \frac{\sqrt{5(a+h)+4} + \sqrt{5a+4}}{\sqrt{5(a+h)+4} + \sqrt{5a+4}} \\
&= \lim_{h\to 0} \frac{5(a+h)+4-(5a+4)}{h\left(\sqrt{5(a+h)+4} + \sqrt{5a+4}\right)} \\
&= \lim_{h\to 0} \frac{5h}{h\left(\sqrt{5(a+h)+4} + \sqrt{5a+4}\right)} \\
&= \frac{5}{2\sqrt{5a+4}}
\end{aligned}$$

Thus, $f'(1) = \dfrac{5}{2\sqrt{5(1)+4}} = \dfrac{5}{6}$.