Be sure that this examination has 16 pages including this cover

**The University of British Columbia**

Final Examinations - December, 2002

**Mathematics 200**

All Sections

Time: 2.5 hours

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| **Name** | ______________ | **Signature** | ______________ |
| **Student Number** | ______________ | **Instructor's Name** | ______________ |
| | | **Section Number** | ______________ |

**Special Instructions:**

Students are allowed to bring into the exam an $8\frac{1}{2}$x11" sheet of paper filled on both sides
No calculators are allowed.

## Rules governing examinations:

1. **Each candidate should be prepared to produce his/her library/AMS card upon request.**

2. **Read and observe the following rules:**
   No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
   Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
   CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
   (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
   (b) Speaking or communicating with other candidates.
   (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. **Smoking is not permitted during examinations.**

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1. The position of a particle at time $t$ (measured in seconds $s$) is given by

$$\mathbf{r}(t) = t\cos(\frac{\pi t}{2})\mathbf{i} + t\sin(\frac{\pi t}{2})\mathbf{j} + t\mathbf{k}$$

(a) Show that the path of the particle lies on the cone $z^2 = x^2 + y^2$. (2)

(b) Find the velocity vector and the speed at time $t$. (4)

(c) Suppose that at time $t = 1s$ the particle flies off the path on a line $L$ in the direction tangent to the path. Find the equation of the line $L$. (3)

(d) How long does it take for the particle to hit the plane $x = -1$ after it started moving along the straight line $L$? (2)

[11 marks]

a) $x^2+y^2 = t^2\cos^2(\frac{\pi t}{2}) + t^2\sin^2(\frac{\pi t}{2}) = t^2 = z^2$

b) $\underset{\sim}{r}' = \left(\cos(\frac{\pi t}{2}) - \frac{\pi t}{2}\sin(\frac{\pi t}{2}),\ \sin(\frac{\pi t}{2}) + \frac{\pi t}{2}\cos(\frac{\pi t}{2}),\ 1\right) = \underset{\sim}{v}$

SPEED $= |\underset{\sim}{v}| = \left\{\left[\cos(\frac{\pi t}{2}) - \frac{\pi t}{2}\sin(\frac{\pi t}{2})\right]^2 + \left[\sin(\frac{\pi t}{2}) + \frac{\pi t}{2}\cos(\frac{\pi t}{2})\right]^2 + 1\right\}^{1/2}$

$= \left\{\cos^2(\frac{\pi t}{2}) - \pi t\sin(\frac{\pi t}{2})\cos(\frac{\pi t}{2}) + (\frac{\pi t}{2})^2\sin^2(\frac{\pi t}{2})\right.$

$\left. + \sin^2(\frac{\pi t}{2}) + \pi t\sin(\frac{\pi t}{2})\cos(\frac{\pi t}{2}) + (\frac{\pi t}{2})^2\cos^2(\frac{\pi t}{2}) + 1\right\}^{1/2}$

$= \left(2 + (\frac{\pi t}{2})^2\right)^{1/2}$

c) $\underset{\sim}{r}'(1) = (0 - \frac{\pi}{2}, 1, 1)$

$\underset{\sim}{r}(1) = (0, 1, 1)$

$\therefore L: \underset{\sim}{r}(\tau) = (0,1,1) + \tau(-\frac{\pi}{2}, 1, 1) = (-\frac{\pi}{2}\tau, 1+\tau, 1+\tau)$

d) $x = -\frac{\pi}{2}\tau = -1$ WHEN $\tau = \frac{2}{\pi}$ S

2. (a) Let $f$ be an arbitrary differentible function defined on the entire real line. Show that the function $w$ defined on the entire plane as

$$w(x,y) = e^{-y} f(x-y)$$

satisfies the partial differential equation:

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

(5)

(b) The equations $x = u^3 - 3uv^2$, and $y = 3u^2v - v^3$ and $z = u^2 - v^2$ define $z$ as a function of $x$ and $y$. Determine $\frac{\partial z}{\partial x}$ at the point $(u, v) = (2, 1)$ which corresponds to the point $(x, y) = (2, 11)$. (10)

[15 marks]

(a) $$\frac{\partial w}{\partial x} = e^{-y} f'(x-y)$$

$$\frac{\partial w}{\partial y} = -e^{-y} f(x-y) - e^{-y} f'(x-y)$$

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = \cancel{e^{-y} f} + \cancel{e^{-y} f'} - \cancel{e^{-y} f} - \cancel{e^{-y} f'} = 0$$

(b) $$x = u^3 - 3uv^2 \quad (1)$$

$$y = 3u^2v - v^3 \quad (2)$$

$$\frac{\partial (1)}{\partial x} \Rightarrow \quad 1 = 3u^2 \frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial x} v^2 - 6uv\frac{\partial v}{\partial x}$$

$$(u,v) = (2,1) \Rightarrow \quad 1 = (12-3)\frac{\partial u}{\partial x} - 12\frac{\partial v}{\partial x} \Rightarrow \quad 9\frac{\partial u}{\partial x} - 12\frac{\partial v}{\partial x} = 1$$

$$\frac{\partial (2)}{\partial x} \Rightarrow \quad 0 = 6u\frac{\partial u}{\partial x} v + 3u^2 \frac{\partial v}{\partial x} - 3v^2 \frac{\partial v}{\partial x}$$

$$(u,v) = (2,1) \Rightarrow \quad 0 = 12\frac{\partial u}{\partial x} + (12-3)\frac{\partial v}{\partial x} \Rightarrow \quad 12\frac{\partial u}{\partial x} + 9\frac{\partial v}{\partial x} = 0$$

$$\begin{bmatrix} 9 & -12 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} 1 & -12 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 9 & -12 \\ 4 & 3 \end{vmatrix}} = \frac{3}{27+48} = \frac{3}{75} = \frac{1}{25}$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} 9 & 1 \\ 4 & 0 \end{vmatrix}}{\begin{vmatrix} 9 & -12 \\ 4 & 3 \end{vmatrix}} = \frac{-4}{75}$$

$$\frac{\partial z}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}\bigg|_{(2,1)} = 4\left(\frac{3}{75}\right) - 2\left(\frac{-4}{75}\right) = \frac{20}{75} = \frac{4}{15}$$

Question 2 (continued)

3. You are standing at a lone palm tree in the middle of the Exponential Desert. The height of the sand dunes around you is given in meters by

$$h(x, y) = 100e^{-(x^2+2y^2)}$$

where $x$ represents the number of meters east of the palm tree (west if $x$ is negative), and $y$ represents the number of meters north of the palm tree (south if $y$ is negative).

(a) Suppose you walk 3 meters east and 2 meters north. At your new location, (3,2), in what direction is the sand dune sloping most steeply downward? (3)

(b) If you walk north from the location described in part (a), what is the instantaneous rate of change of height of the sand dune? (2)

(c) If you are standing at (3,2) in what direction should you walk to ensure that you remain at the same height? (2)

(d) Find the equation of the curve through (3,2) that should you move along in order that you are always pointing in a steepest descent direction at each point of this curve. (5)

[12 marks]

a) $\nabla h = 100 e^{-(x^2+2y^2)}(-2x, -4y)$

THE STEEPEST DESCENT DIRECTION IS $-\underset{\sim}{\nabla} h(3,2) = 100e^{-(9+8)}(2.3, 4.2)$

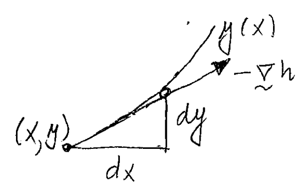
$= 100 e^{-17}(6,8) = 200e^{-17}(3,4)$

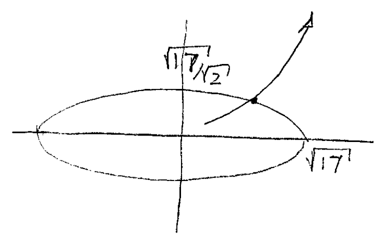
b) $\underset{\sim}{\hat{N}} = (0,1)$ $\quad D_{\hat{N}} h = \underset{\sim}{\hat{N}} \cdot \underset{\sim}{\nabla} h = (0,1) \cdot \underset{\sim}{\nabla} h(3,2) = -800 e^{-17}$

c) TO WALK ALONG A LEVEL CURVE HEAD IN A DIRECTION $\perp$ TO $\underset{\sim}{\nabla} h$

SO $(-4,3)$ OR $(4,-3)$.

5 d)




$-\underset{\sim}{\nabla} h \parallel (dx, dy)$

$-\underset{\sim}{\nabla} h = k(dx, dy)$

$\therefore \ 100 e^{-(x^2+2y^2)}(2x, 4y) = k(dx, dy)$

$\dfrac{k\,dy}{k\,dx} = \dfrac{100 e^{-(x^2+2y^2)} \cdot 4y}{100 e^{-(x^2+2y^2)} 2x} \Rightarrow \dfrac{dy}{dx} = \dfrac{2y}{x}$ $\quad$ SEPARABLE ODE

$\therefore \int \dfrac{dy}{y} = 2\int \dfrac{dx}{x} + C$

$\therefore \ \ln|y| = 2\ln|x| + C$

$y = Ax^2$

$y(3) = 2 \Rightarrow 2 = A.3^2 \Rightarrow A = \dfrac{2}{9}$

$\therefore \boxed{y = \dfrac{2}{9}x^2}$

Question 3 (continued)

4. Find all the critical points of the function

$$f(x,y) = x^4 + y^4 - 4xy$$

defined in the $x$-$y$ plane. Classify each critical point as a local minimum, maximum, or saddle point. Explain your reasoning.

[12 marks]

LOOK FOR ALL CRITICAL POINTS:

2 $f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3$
$f_y = 4y^3 - 4x = 0 \Rightarrow x = y^3$ $\Big\} \Rightarrow x = (x^3)^3 = x^9$ ; $x(x^8 - 1) = 0$

$\therefore$ $(x,y) = (0,0)$, $(-1,-1)$ $(1,1)$ ARE THE CRITICAL POINTS. 3

NOW CHECK THESE CP FOR LOCAL MAX/MIN/SADDLE

$f_{xx} = 12x^2$
$f_{yy} = 12y^2$
$f_{xy} = -4$ $\Big\}$ $D(x,y) = 144x^2y^2 - (-4)^2 = 144x^2y^2 - 16$ 2

(I) CP: $(0,0)$: $D(0,0) = -16$ SO THE CP IS A SADDLE POINT 2
THE VALUE AT THIS POINT IS $f(0,0) = 0$

(II) CP $(\pm1,\pm1)$: $D(\pm1,\pm1) = 144 - 16 = 128 > 0$
$f_{xx}(\pm1,\pm1) = 12 > 0$ $\Big\} \Rightarrow (\pm1,\pm1)$ ARE LOCAL MINIMA. 2

THE VALUES AT THE LOCAL MINIMA ARE $f(\pm1,\pm1) = 1 + 1 - 4 = -2$.

5. (a) By finding the points of tangency determine the values of $c$ for which $x + y + z = c$ is a tangent plane to the surface $4x^2 + 4y^2 + z^2 = 96$.

(b) Use the method of Lagrange Multipliers to determine the absolute maximum and minimum values of the function $f(x, y, z) = x + y + z$ along the surface $g(x, y, z) = 4x^2 + 4y^2 + z^2 = 96$.

(c) Why do you get the same answers in (a) and (b)?

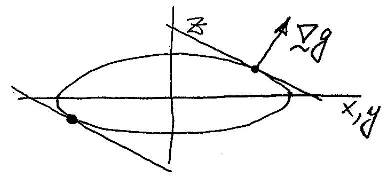
[12 marks]

[12 marks]

a) $\underset{\sim}{\nabla} g = (8x, 8y, 2z) \parallel (1,1,1)$

$(8x, 8y, 2z) = k(1,1,1)$

$k = 8x = 8y = 2z.$




$\therefore\ x = y \qquad z = 4x.$ BUT POINT MUST BE ON THE SURFACE

$\therefore\ 4x^2 + 4x^2 + (4x)^2 = 96$

$24x^2 = 96$

$x = \pm 2 = y \qquad z = \pm 8$

$(x, y, z) = (2, 2, 8)$ OR $(-2, -2, -8)$ [5]

SO THAT $c = 2 + 2 + 8 = 12$ OR $-12$.

b) $\left.\begin{array}{l} f_x = 1 = \lambda g_x = \lambda 8x \\ f_y = 1 = \lambda g_y = \lambda 8y \\ f_z = 1 = \lambda g_z = \lambda 2z \end{array}\right\}\quad \frac{1}{\lambda} = 8x = 8y = 2z$

$x = y \qquad z = 4x.$

$\therefore\ g = 96 \Rightarrow 4x^2 + 4x^2 + (4x)^2 = 24x^2 = 96 \Rightarrow x = \pm 2.$

$\therefore\ (x, y, z) = (2, 2, 8)$ OR $(-2, -2, -8)$ [5]

SO THE MAX/MIN VALUES OF $f$ ON THE ELLIPSOID ARE 12 & −12

c) THE METHOD OF LAGRANGE MULTIPLIERS IS BASED ON LOCATING THE POINTS OF TANGENCY OF $f =$ CONST AND THE CONSTRAINT SURFACE $g = k$. THIS YIELDS THE CONDITION [2]

$\underset{\sim}{\nabla} f = \lambda \underset{\sim}{\nabla} g.$

6. Evaluate the following integral:

$$I = \int_{-2}^{2}\int_{x^2}^{4} \cos(y^{\frac{3}{2}})\,dy\,dx$$

[8 marks]

$$I = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \cos(y^{3/2})\,dx\,dy$$

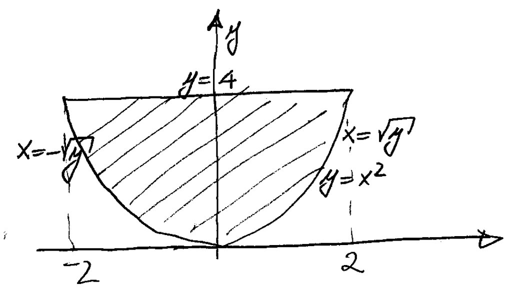
$$= \int_0^4 \left[\cos(y^{3/2})x\right]_{-\sqrt{y}}^{\sqrt{y}} dy$$

$$= 2\int_0^4 \cos(y^{3/2})\,y^{1/2}\,dy$$

$$= \frac{4}{3}\int_0^8 \cos u\,du$$

$$= \frac{4}{3}\left[\sin u\right]_0^8$$

$$= \frac{4}{3}\sin 8$$




LET $u = y^{3/2}$ $du = \frac{3}{2}y^{1/2}dy$

$y = 0 \Rightarrow u = 0$ & $y = 4 \Rightarrow u = 8$

7. Let $D$ be the region in the $xy$-plane which is inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 2$. Determine the mass of this region if the density is given by:

$$\rho(x,y) = \frac{2}{\sqrt{x^2+y^2}}$$

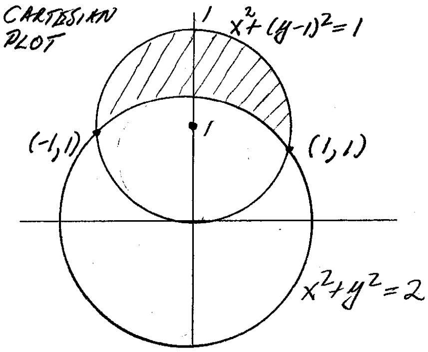
[15 marks]

INTERSECTION PTS:

$$x^2+y^2-2y+1=1 \quad \& \quad x^2+y^2=2$$

$$2(1-y)=0 \qquad y=1 \ \& \ x=\pm 1$$

CARTESIAN PLOT




CONVERT TO POLAR COORDINATES

$$x^2+y^2-2y=0$$

$$\Rightarrow \quad r^2 = 2r\sin\theta$$

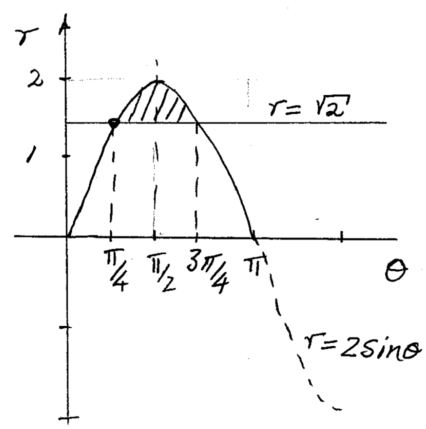
$$\boxed{r = 2\sin\theta}$$

$$x^2+y^2=2 \Rightarrow \boxed{r=\sqrt{2}}$$

INTERSECTION POINTS

$$r=\sqrt{2} = 2\sin\theta \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4,\ 3\pi/4$$

$r$-$\theta$ PLOT




$$M = \iint_D \frac{2}{\sqrt{x^2+y^2}}\,dA$$

$$= \int_{\pi/4}^{3\pi/4}\int_{\sqrt{2}}^{2\sin\theta} \frac{2}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$$

$$= 2\int_{\pi/4}^{3\pi/4} (2\sin\theta - \sqrt{2})\,d\theta$$

$$= 2\left[-2\cos\theta - \sqrt{2}\,\theta\right]_{\pi/4}^{3\pi/4}$$

$$= 2\left[-2\left(\frac{-1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\right]$$

$$= 2\left[2\sqrt{2} - \frac{\pi}{\sqrt{2}}\right]$$

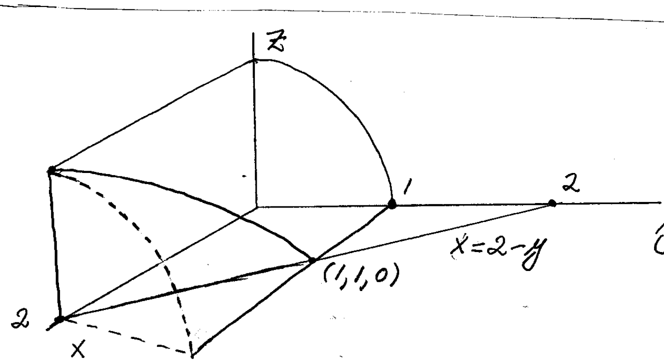
$$= 4\sqrt{2} - \sqrt{2}\,\pi$$

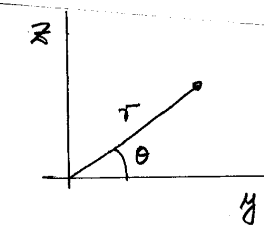
$$= (4-\pi)\sqrt{2}$$

Question 7 (continued)

8. Evaluate $\iiint\limits_E z\,dV$, where $E$ is the region bounded by the planes $y = 0$, $z = 0$, $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant.

[15 marks]




$y = r\cos\theta \quad z = r\sin\theta.$

SET UP POLAR COORDINATES IN THE $y$-$z$ PLANE AS SHOWN: TO DEFINE THE CYLINDRICAL COORDINATES $(r, \theta, x)$.

$$I = \iiint\limits_E z\,dV$$

$$= \int_0^{\pi/2}\int_0^1\int_0^{(2-r\cos\theta)} r\sin\theta\,dx\;r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \Big[(r\sin\theta)\,x\Big]_0^{2-r\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r\sin\theta\,(2-r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 2r^2\sin\theta - r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\Big[\frac{2r^3}{3}\sin\theta - \frac{r^4}{4}\cos\theta\sin\theta\Big]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{2}{3}\sin\theta - \frac{1}{8}\sin 2\theta\,d\theta$$

$$= \Big[-\frac{2}{3}\cos\theta + \frac{1}{16}\cos 2\theta\Big]_0^{\pi/2}$$

$$= \Big[-\frac{2}{3}\overset{0}{\cancel{\cos\tfrac{\pi}{2}}} + \frac{2}{3}\cos 0 + \frac{1}{16}\big(\cos\pi - \cos(0)\big)\Big]$$

$$= \Big[\frac{2}{3} - \frac{1}{8}\Big]$$

$$= \frac{13}{24}$$

Question 8 (continued)

Additional workspace

Additional workspace