Math 1229A/B

Unit 1:

Vectors

(text reference: Section 1.1)

1 Vectors

You are familiar with the set of real numbers. **Real numbers** means all the numbers you've ever heard of or can imagine (unless you've learnt about, or at least heard of, *imaginary* numbers – they're not *real* numbers). Real numbers includes all the integers (including both positive and negative, and of course 0), fractions (called *rational numbers*), decimals that can't be expressed as fractions (the *irrational numbers*). All the numbers along the real number line from $-\infty$ to ∞. We call the set of real numbers $\Re$.

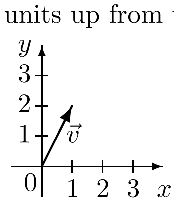
You're also familiar with the x-y plane. You've drawn graphs in this plane. There's the x-axis (horizontal) and the y-axis (vertical), which cross at the *origin*. Each axis is basically a real number line. Any point in this plane can be expressed as an ordered pair, (x, y), giving its x-*coordinate* and its y-*coordinate*. Each coordinate can be any real number. We call the x-y plane **2-space** and the set containing all of the points in this plane is called $\Re^2$. (We pronounce that "R2", i.e. "Artoo".) So $\Re^2$ can be thought of as the set of all ordered pairs of real numbers.

You've probably also seen something even more complicated, with 3 axes: the x-axis, the y-axis and the z-axis. Each axis is perpendicular to both of the others, which makes it hard to draw on a piece of paper or a blackboard. They're often drawn with the y-axis horizontal, the z-axis vertical, and the x-axis off at a funny angle, to represent that it's coming straight out of the page at you. The 3 axes represent the 3 dimensions of "space", i.e. reality. Like the room you're sitting in. There's not only up and down, and left and right, but also near and far, or here and there, or ... well, you know, that third dimension, which we might call depth. In Math, we call the region defined by these 3 axes **3-space**. And points in 3-space are represented by *ordered triples*, (x, y, z), giving the x-, y- and z-coordinates of the point. As before, each of these coordinates can be any real number. The set containing all of the points in 3-space is called $\Re^3$ (pronounced "R3"), and we can think of this as the set of all ordered triples of real numbers.

Now that you've got that straight, let's confuse things. Sometimes when we write an ordered pair or an ordered triple, it doesn't represent a point. Instead, it represents a *directed line segment*, called a **vector**. And when the pair or triple represents a vector, the numbers in it (or symbols representing numbers) aren't called coordinates, they're called **components**.

That seems like it will be confusing, using the same notation to denote two different things. It's not too bad, though, because you can tell by the *context* whether a pair or triplet is a point or a vector. And because we write the *names* differently, depending whether it's a point or a vector. Points are named with normal capital letters, usually P or something nearby in the alphabet, and sometimes with subscripts. So we might have the points $P(p_1, p_2)$ and $Q(q_1, q_2)$, or the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. With vectors, we don't use capital letters, and we do something to show that it's a vector. In our textbook, they use **boldface** type for the name of a vector. For instance, the vector $\mathbf{v} = (v_1, v_2)$ or the vector $\mathbf{u} = (u_1, u_2, u_3)$. "Sure", I hear you thinking, "that's easy enough for you, but what about me, when I'm writing with a pen or pencil?". Well, that's why we're going to use a different convention in these notes. One that's much more obvious. When a letter is the name of a vector, we'll put an arrow over it. Like this: $\vec{v} = (v_1, v_2)$.

So why is it that we represent a vector, which we said is a directed line segment, using something that looks like a point? Well, it's just a convention. It's shorthand. When we say $\vec{v} = (1, 2)$, what we mean is that the vector $\vec{v}$ is the directed line segment that starts at the origin and ends at the point $(1, 2)$. So here's a picture of the vector $\vec{v} = (1, 2)$. It starts at the point $(0, 0)$, and then ends at a place that's 1 unit to the right and 2 units up from there.



> **Definition:** The **vector** $\vec{v} = (v_1, v_2)$ **in** $\Re^2$ is the directed line segment that goes from the origin (i.e. the point $(0, 0)$) to the point $V(v_1, v_2)$. Similarly, the **vector** $\vec{v} = (v_1, v_2, v_3)$ **in** $\Re^3$ is the directed line segment that goes from the point $(0, 0, 0)$ to the point $V(v_1, v_2, v_3)$. The point V is called the **endpoint** of the vector $\vec{v}$.

If we want to say that $\vec{v}$ is a vector in $\Re^2$, i.e. a vector which has two components, then we say $\vec{v} \in \Re^2$. You've probably seen that symbol before, for instance for saying that a particular object is an element of a particular set. Similarly, we can say $\vec{v} \in \Re^3$ to state that $\vec{v}$ is a vector in $\Re^3$. Notice that earlier we said that $\Re^2$ was the set of all points in the x-y plane. But now we're saying that a vector is in that set. Hmm. If $\Re^2$ is a set, does it contain points or vectors? Well, actually, we can think of it either way. We can describe 2-space as the set of all points in the x-y plane, or as the set of all vectors in the x-y plane. Or we can define it simply as the set of all ordered pairs (x, y). That's probably the best way to think of it. Because an ordered pair can (as we've already discussed) represent either a point or a vector, depending on the context. Likewise, we'll think of $\Re^3$ as simply the set of all ordered triples (x, y, z).

There is a vector in $\Re^2$ corresponding to each point in the x-y plane. Likewise, there is a vector in $\Re^3$ corresponding to each point in 3-space. But wait a minute! What about the point (0,0) or (0,0,0)? That's where the directed line segment starts. So can there be a directed line segment that goes from that point to itself? Well, yes. Although you won't be able to see it, and you won't care, or be able to determine, what direction it goes in. That is, in spite of the fact that "the line segment from the point (0,0) to the point (0,0)" seems nonsensical, because there's no line segment there, we *do* consider there to be a vector $\vec{v} = (0, 0)$. It actually comes in very handy. We call it the "zero vector", and give it the name $\vec{0}$.

> **Definition:** A **zero vector** is a vector whose components are all 0. The **zero vector in** $\Re^2$ is the vector $\vec{0} = (0, 0)$. Similarly, $\vec{0} = (0, 0, 0)$ is the **zero vector in** $\Re^3$.

Whenever we define a new mathematical construct, we need to define what "equality" means for that construct. Even if it seems pretty obvious. So we need to define what it means to say that two vectors are equal. We've already used that concept, in attaching names to vectors. For instance when we say $\vec{v} = (v_1, v_2)$, we're saying that the vector whose name is $\vec{v}$ is equal to the vector in $\Re^2$ whose first component is v_1 and whose second component is v_2. Likewise, when we say $\vec{0} = (0, 0, 0)$, we're saying that the vector whose name is $\vec{0}$ is equal to the vector in $\Re^3$ whose components are all 0. But of course, what we really meant there is "this is the name I'm going to call that vector by", rather than "here are 2 different vectors, and they're equal". But often we do need to equate 2 vectors in that sense, too. Or to say that the vector you get when you do certain vector arithmetic operations (which we'll learn about shortly) is equal to a specified vector. So what do we mean when we say, for instance, that $\vec{u} = \vec{v}$?

> **Definition:** Two vectors are **equal** if they are vectors in the same space and their corresponding components are equal. That is, two vectors in $\Re^2$ are equal if they have the same first component and also have the same second component. Similarly, two vectors in $\Re^3$ are equal if they have the same first component and have the same second component and have the same third component. In mathematical notation, we have
>
> If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, then $\vec{u} = \vec{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$.
>
> and similarly
>
> If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} = \vec{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$ and $u_3 = v_3$.

Notice that in order for 2 vectors to be equal they *must* be vectors in the same space. If $\vec{u} \in \Re^2$ and $\vec{v} \in \Re^3$ then they can **never** be equal vectors, no matter what their components are.

Example 1.1. If $\vec{u} = (a, 2)$ and $\vec{v} = (-1, b)$, where it is known that $\vec{u} = \vec{v}$, what are the values of a and b?

Solution:
Since $\vec{u} = \vec{v}$, then $(a, 2) = (-1, b)$. And for these vectors to be equal, their respective components must be equal. Since the first component of $\vec{v}$ is -1 and $\vec{u} = \vec{v}$, then the first component of $\vec{u}$ must also be -1. And a is the first component of $\vec{u}$, so it must be true that $a = -1$. Likewise, since the second component of $\vec{u}$ is 2 and $\vec{v} = \vec{u}$, then the second component of $\vec{v}$ must also be 2. But the second component of $\vec{v}$ is b, and so $b = 2$.

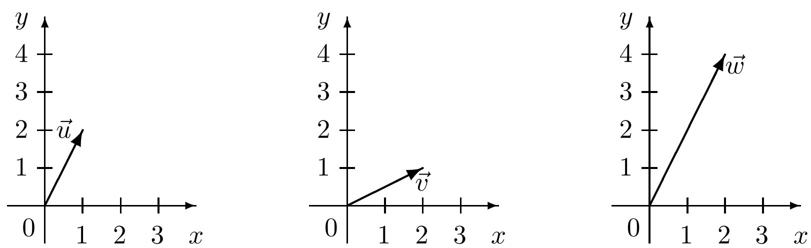
All vectors start at the origin. There are infinitely many lines that pass through the origin. For any vector (other than the zero vector), there's exactly one line through the origin that the vector lies on. And often it's important to realize whether or not 2 vectors lie on the same line. We have a word for that. **Collinear** just means "same line".

Definition: Two vectors in the same space are **collinear** if they lie on the same line.

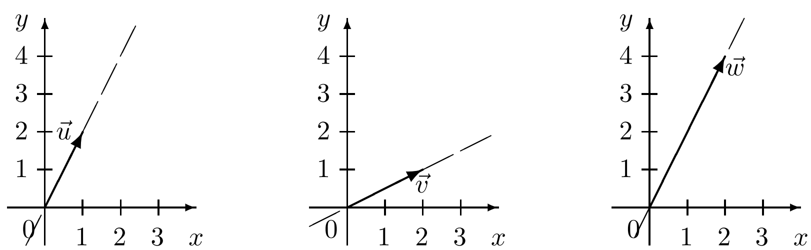
Of course, if two vectors lie on the same line, they must be parallel to one another. So if two vectors are collinear, they are also parallel, and vice versa.

Example 1.2. Consider the vectors $\vec{u} = (1, 2)$, $\vec{v} = (2, 1)$ and $\vec{w} = (2, 4)$. Draw each of these vectors. Show that $\vec{u}$ and $\vec{v}$ are not collinear, but that $\vec{u}$ and $\vec{w}$ are.

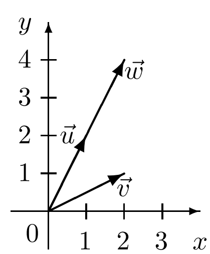
Solution:
First we'll draw each vector on its own set of axes.



We can also show the line that each vector lies on:



It's pretty clear that the line that $\vec{u}$ lies on is not the same as the line that $\vec{v}$ lies on. For $\vec{u}$ and $\vec{w}$, they look pretty much the same, but how can we be sure? We draw them all on the same axes:



From this last diagram, even without the lines drawn in we can see that vectors $\vec{u}$ and $\vec{v}$ are certainly not collinear, and also that vector $\vec{w}$ lies right on top of $\vec{u}$, because they *are* collinear.

All vectors start at the origin, so all vectors (in the same space) touch one another. And yet, we talk about the *distance* between vectors. By this, we mean the furthest distance between any 2 points with one point being on each vector. This always occurs at the endpoints of the vectors.

Definition: The **distance between** two vectors $\vec{u}$ and $\vec{v}$ is defined to be the *distance between their endpoints* and is denoted $d(\vec{u}, \vec{v})$. In $\Re^2$ we have:

$$d(\vec{u}, \vec{v}) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$$

Similarly, in $\Re^3$ we have:

$$d(\vec{u}, \vec{v}) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}$$

The formula for the distance between two vectors in $\Re^2$ is just an application of the Pythagorean Theorem, found by considering the line segment joining the two endpoints to be the hypotenuse of a right-angled triangle. The height of the triangle is the vertical distance between the two points (i.e. the difference between their y-coordinates, or the second components of the vectors) and likewise the length of the base of the triangle is the horizontal distance between the two points (i.e. the difference between their x-coordinates, or the first components of the vectors). The formula for the distance between two vectors in $\Re^3$ is based on the same idea, but in 3 dimensions. Notice that because the terms inside the square root are each squared, the distance between $\vec{u}$ and $\vec{v}$ is the same as the distance between $\vec{v}$ and $\vec{u}$. That is, $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$. So it doesn't matter which vector is mentioned first.

Example 1.3. For the vectors $\vec{u} = (1, 2)$, $\vec{v} = (2, 1)$ and $\vec{w} = (2, 4)$, find
(a) $d(\vec{u}, \vec{v})$ (b) $d(\vec{w}, \vec{u})$

Solution:
(a) We have $u_1 = 1$, $u_2 = 2$, $v_1 = 2$ and $v_2 = 1$. (That is, when we refer to the components of $\vec{u}$ as u_1 and u_2, we simply mean whatever numbers are the first and second components, respectively, of the vector. Similarly for a vector with three components.) So for the distance between $\vec{u}$ and $\vec{v}$ we get

$$d(\vec{u}, \vec{v}) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2} = \sqrt{(2-1)^2 + (1-2)^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

(b) For the distance between $\vec{w}$ and $\vec{u}$ we do a similar calculation:

$$d(\vec{w}, \vec{u}) = \sqrt{(u_1 - w_1)^2 + (u_2 - w_2)^2} = \sqrt{(1-2)^2 + (2-4)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$$

Notice that the formula says to take, for each component, the square of the *second-mentioned* vector minus the *first-mentioned* vector. So even though the formula said $(v_1 - u_1)^2$, in calculating $d(\vec{w}, \vec{u})$, we put u_1 before the minus sign, not after. (That is, in this calculation, $\vec{u}$ was filling the role played by $\vec{v}$ in the formula.) In this particular formula, it doesn't matter because, as previously mentioned, the distance is the same, whether we think of it as the distance between $\vec{w}$ and $\vec{u}$ or as the distance between $\vec{u}$ and $\vec{w}$. (Is the distance between *here* and *there* any different than the distance between *there* and *here*? Well, I suppose sometimes, when there are one-way streets involved. But not usually.) In other situations, though, it will be important to use the right vector in the right place in the formula.

Example 1.4. Find the distance between $\vec{u} = (1, 2, 3)$ and $\vec{v} = (-1, 0, 1)$.

Solution:
We do the same sort of calculation as before, but now with a third component. We get:

$$\begin{aligned} d(\vec{u}, \vec{v}) &= \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2} = \sqrt{(-1-1)^2 + (0-2)^2 + (1-3)^2} \\ &= \sqrt{(-2)^2 + (-2)^2 + (-2)^2} = \sqrt{4+4+4} = \sqrt{4 \times 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3} \end{aligned}$$

The length of a vector is defined as the distance between where it starts and where it ends. That is, the distance from the origin (where all vectors start) to the endpoint of the vector. There are some other words we sometimes use which mean exactly the same thing. We sometimes talk about the *magnitude* or the *norm* of a vector. These terms both mean the length of the vector.

> **Definition:** The **length** of a vector $\vec{v}$, also called the **magnitude** or the **norm** of the vector, is denoted by $||\vec{v}||$ and is defined to be
>
> $$||\vec{v}|| = d(0, \vec{v})$$
>
> Therefore for $\vec{v} \in \Re^2$ we have:
>
> $$||\vec{v}|| = \sqrt{(v_1)^2 + (v_2)^2}$$
>
> and for $\vec{v} \in \Re^3$ we have:
>
> $$||\vec{v}|| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$$
>
> Any vector whose length is 1 is called a **unit vector**.

Notice that the only way to get $||\vec{v}|| = 0$ is by having each component of $\vec{v}$ be 0. So $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$.

Example 1.5. Find the length of $\vec{u} = (1, 2)$, the magnitude of $\vec{v} = (2, 1)$ and the norm of $\vec{w} = (2, 4)$.

Solution:
Length, magnitude and norm all mean the same thing, so we just use the same formula three times, once for each vector. We get

$$\begin{aligned}
||\vec{u}|| &= \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5} \\
||\vec{v}|| &= \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5} \\
||\vec{w}|| &= \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20}
\end{aligned}$$

Notice that $\sqrt{20} = \sqrt{(4)(5)} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$. The length of $\vec{w}$ is twice the length of $\vec{u}$. That is, $||\vec{w}|| = 2||\vec{u}||$.

Example 1.6. Find $\left|\left|\left(\frac{3}{5}, 0, -\frac{4}{5}\right)\right|\right|$.

Solution:

$$\begin{aligned}
\left|\left|\left(\frac{3}{5}, 0, -\frac{4}{5}\right)\right|\right| &= \sqrt{\left(\frac{3}{5}\right)^2 + (0)^2 + \left(-\frac{4}{5}\right)^2} \\
&= \sqrt{\left(\frac{3^2}{5^2}\right) + 0 + \left(\frac{(-4)^2}{5^2}\right)} \\
&= \sqrt{\frac{9}{25} + \frac{16}{25}} \\
&= \sqrt{\frac{9 + 16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1
\end{aligned}$$

Notice: Since $\left|\left|\left(\frac{3}{5}, 0, -\frac{4}{5}\right)\right|\right| = 1$ then $\left(\frac{3}{5}, 0, -\frac{4}{5}\right)$ is a unit vector.

Example 1.7. If $\vec{u} = \left(\frac{2}{3}, \frac{2}{3}, k\right)$ is a unit vector, what is the value of k?

Solution:
We need $||\vec{u}|| = 1$ in order for $\vec{u}$ to be a unit vector. Since $||\vec{u}||$ is given by

$$||\vec{u}|| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + k^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + k^2} = \sqrt{\frac{8}{9} + k^2}$$

then we must have

$$\sqrt{\frac{8}{9} + k^2} = 1$$

$$\text{so} \qquad \left(\sqrt{\frac{8}{9} + k^2}\right)^2 = 1^2 \quad \rightarrow \quad \frac{8}{9} + k^2 = 1$$

$$\text{therefore} \quad k^2 = 1 - \frac{8}{9} = \frac{9}{9} - \frac{8}{9} = \frac{9-8}{9} = \frac{1}{9}$$

$$\text{and so} \qquad k = \pm\sqrt{\frac{1}{9}} = \pm\frac{\sqrt{1}}{\sqrt{9}} = \pm\frac{1}{3}$$

(That is, since both $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ and $\left(-\frac{1}{3}\right)^2 = \frac{1}{9}$, then knowing that $k^2 = \frac{1}{9}$ tells us that k is one of these 2 values, but doesn't tell us which one it is.)

We see that k could be either $\frac{1}{3}$ or $-\frac{1}{3}$. That is, both $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ and $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ are unit vectors, and $\vec{u}$ could be either of these vectors.

Definition: A **scalar** is just a number. Thus any element of $\Re$ is a scalar.

A vector can be multiplied, or scaled, by a number. When a vector (or other entity, other than simply a number) is multiplied by a scalar, we call it *scalar multiplication*. When a vector is multiplied by a scalar, the effect is that *each component* of the vector is multiplied by that scalar.

Definition: Let $c \in \Re$ be any scalar and $\vec{v} \in \Re^2$ or $\Re^3$ be any vector. The **scalar multiple** $c\vec{v}$ is obtained by:

$$c\vec{v} = (cv_1, cv_2) \text{ if } \vec{v} \in \Re^2, \text{ or } c\vec{v} = (cv_1, cv_2, cv_3) \text{ if } \vec{v} \in \Re^3$$

For instance, for the vectors in Examples 1.2 and 1.3 we had

$$\vec{w} = (2, 4) = (2 \times 1, 2 \times 2) = 2(1, 2) = 2\vec{u}$$

The vector $\vec{w}$ is two times the vector $\vec{u}$, which is why we found that it was twice as long, i.e. has length twice the length of $\vec{u}$.

Every scalar multiple of a vector lies along the same line as that vector. And from the origin, there are (only) two directions you can go along a line that passes through the origin. So there are two directions which a scalar multiple of a vector could go: the same direction as the vector, or the *opposite* direction. For instance, if $\vec{v}$ goes straight up, then some scalar multiples of $\vec{v}$ go straight up, and others go straight down. Or if the scalar is 0, then the scalar multiple doesn't go anywhere at all. Scalars bigger than 0 give vectors with the same direction, while scalars less than 0 give vectors with the opposite direction. And whether a new vector goes twice as far as $\vec{v}$ in the same direction as $\vec{v}$, or goes twice as far as $\vec{v}$ in the opposite direction, it will still be true that the new vector is twice as long as $\vec{v}$. That is, both $2\vec{v}$ and $-2\vec{v}$ have magnitude $2||\vec{v}||$, but $2\vec{v}$ goes in the same direction as $\vec{v}$, while $-2\vec{v}$ goes in the opposite direction.

Theorem 1.1. *Let $\vec{v}$ be any vector, either in $\Re^2$ or $\Re^3$, and consider any $c \in \Re$. The vectors $\vec{v}$ and $c\vec{v}$ are collinear, and*

1. *if $c > 0$ then $c\vec{v}$ has the same direction as $\vec{v}$;*
2. *if $c < 0$ then $c\vec{v}$ has the opposite direction to $\vec{v}$.*

Also, $||c\vec{v}|| = |c|\,||\vec{v}||$.

Saying that two vectors are collinear is the same as saying that they are parallel. So $\vec{v}$ and $c\vec{v}$ are always parallel to one another, no matter what the value of the scalar c is. And the last part of the theorem says that you can find the magnitude of the scalar multiple of a vector by multiplying the magnitude of the vector by the *absolute value* of the scalar. So for instance, as observed before, both $2\vec{v}$ and $-2\vec{v}$ have magnitude $2||\vec{v}||$.

Notice that the zero vector is collinear to every vector, because for any $\vec{v}$ we have $0\vec{v} = \vec{0}$, so the zero vector is a scalar multiple of every vector (with the same number of components).

Example 1.8. Let $\vec{u} = (2, -3)$ and $\vec{v} = (0, -1, 2)$. Find the vectors $2\vec{u}$, $-0.5\vec{u}$, $-3\vec{v}$ and $\frac{10}{3}\vec{v}$, and find the magnitude of each of these vectors.

Solution:

$$\begin{aligned}
2\vec{u} &= 2(2,-3) = (2\times 2, 2\times(-3)) = (4,-6) \\
-0.5\vec{u} &= -0.5(2,-3) = ((-0.5)\times 2, (-0.5)\times(-3)) = (-1, 1.5) \\
-3\vec{v} &= -3(0,-1,2) = ((-3)(0), (-3)(-1), (-3)(2)) = (0, 3, -6) \\
\frac{10}{3}\vec{v} &= \frac{10}{3}(0,-1,2) = \left(\frac{10}{3}\times 0, \frac{10}{3}\times(-1), \frac{10}{3}\times 2\right) = \left(0, -\frac{10}{3}, \frac{20}{3}\right)
\end{aligned}$$

We could find the magnitudes of these new vectors using the appropriate formula for each. But it's easier to just find the magnitudes of $\vec{u}$ and $\vec{v}$ and use those to find the magnitudes of the given vectors.

$$\begin{aligned}
||\vec{u}|| &= ||(2,-3)|| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13} \\
\text{so } ||2\vec{u}|| &= |2|\,||\vec{u}|| = 2\sqrt{13} \\
\text{and } ||-0.5\vec{u}|| &= |-0.5|\,||\vec{u}|| = 0.5\sqrt{13} = \frac{\sqrt{13}}{2} = \frac{\sqrt{13}}{\sqrt{4}} = \sqrt{\frac{13}{4}} = \sqrt{3.25} \\
\text{Also } ||\vec{v}|| &= ||(0,-1,2)|| = \sqrt{(0)^2 + (-1)^2 + (2)^2} = \sqrt{0+1+4} = \sqrt{5} \\
\text{so } ||-3\vec{v}|| &= |-3|\,||\vec{v}|| = 3\sqrt{5} \\
\text{and } \left|\left|\frac{10}{3}\vec{v}\right|\right| &= \left|\frac{10}{3}\right|\,||\vec{v}|| = \frac{10}{3}\sqrt{5} = \frac{10\sqrt{5}}{3}
\end{aligned}$$

Example 1.9. Let $\vec{u} = (4, -3)$ and $\vec{v} = (5, 0, -2)$. Find a unit vector in the same direction as $\vec{u}$, and a unit vector in the opposite direction to $\vec{v}$.

Solution:
The magnitude of $\vec{u}$ is

$$||\vec{u}|| = ||(4, -3)|| = \sqrt{(4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

We get a vector in the same direction as $\vec{u}$ by taking a scalar multiple of $\vec{u}$, using a positive scalar. Let c be the positive scalar for which $c\vec{u}$ is a unit vector. Then we need $||c\vec{u}|| = 1$. But we know that $||c\vec{u}|| = |c|\,||\vec{u}||$, where in this case $||\vec{u}|| = 5$, and since c is positive, then $|c| = c$. So we need

$$c||\vec{u}|| = 1 \qquad \Rightarrow \qquad 5c = 1 \qquad \Rightarrow \qquad c = \frac{1}{5}$$

Therefore, a unit vector in the same direction as $\vec{u}$ is the vector

$$c\vec{u} = \frac{1}{5}(4, -3) = \left(\frac{4}{5}, -\frac{3}{5}\right)$$

Notice: For any vector $\vec{u}$, if $c\vec{u}$ is a unit vector, then $|c| = \dfrac{1}{||\vec{u}||}$.

We take a similar approach for finding a unit vector in the opposite direction to $\vec{v}$. As we've already seen, for a unit vector we need to scale $\vec{v}$ by a constant whose magnitude (i.e. absolute value) is $\frac{1}{||\vec{v}||}$. And for the vector to have the opposite direction to that of $\vec{v}$, the constant must be negative. So to get a unit vector in the opposite direction to $\vec{v}$, we multiply $\vec{v}$ by the scalar $-\frac{1}{||\vec{v}||}$. We have

$$||\vec{v}|| = ||(5, 0, -2)|| = \sqrt{(5)^2 + (0)^2 + (-2)^2} = \sqrt{25 + 0 + 4} = \sqrt{29}$$

and so a unit vector in the opposite direction to $\vec{v}$ is the vector $c\vec{v}$ with $c = -\frac{1}{||\vec{v}||}$, which gives

$$-\frac{1}{||\vec{v}||}\vec{v} = -\frac{1}{\sqrt{29}}(5, 0, -2) = \left(-\frac{5}{\sqrt{29}}, 0, \frac{2}{\sqrt{29}}\right)$$

In Theorem 1.1 we observed that for any vector $\vec{v}$ and any scalar c, the vectors $\vec{v}$ and $c\vec{v}$ are collinear. But it is also true that if 2 vectors are collinear then it *must* be true that they are scalar multiples of one another.

Theorem 1.2. *Vectors $\vec{u}$ and $\vec{v}$ are collinear if and only if there is some scalar value c such that $\vec{v} = c\vec{u}$.*

Example 1.10. Let $\vec{u} = (-2, 7)$ and $\vec{v} = (4, k)$. If it is known that $\vec{u}$ and $\vec{v}$ are collinear, what is the value of k?

Solution:
We know that if 2 vectors are collinear, one can be written as a scalar multiple of the other. So knowing that $\vec{u}$ and $\vec{v}$ are collinear tells us that there is some value c for which $\vec{v} = c\vec{u}$. We have

$$c\vec{u} = c(-2, 7) = (-2c, 7c)$$

Therefore to have $c\vec{u} = \vec{v}$, we need $(-2c, 7c) = (4, k)$. And of course these vectors are only equal if their corresponding components are the same. So we must have $-2c = 4$ and $7c = k$. We use the first of these to solve for c, and then substitute that in to find k. We get

$$-2c = 4 \rightarrow c = \frac{4}{-2} = -2 \rightarrow k = 7c = 7(-2) = -14$$

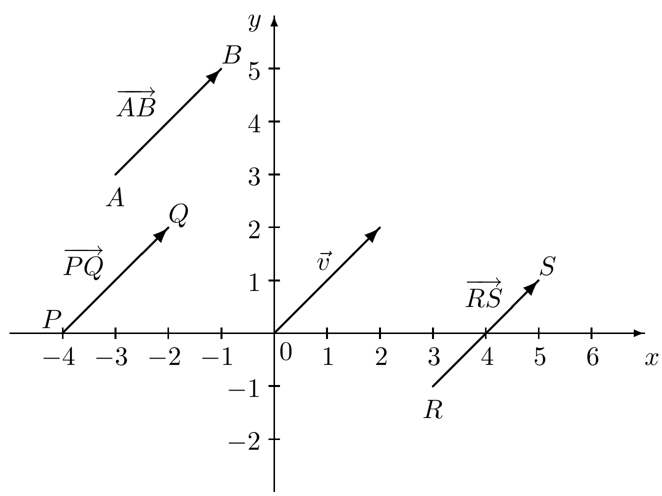
The only vector $(4, k)$ which is collinear with $\vec{u}$ is the vector $(4, -14)$.

Translation of Vectors

A vector is a directed line segment which starts at the origin. A directed line segment from some point P to some point Q, where P is not the origin, is not called a vector. (So all vectors are directed line segments, but only some directed line segments are vectors.) We use $\overrightarrow{PQ}$ to denote such a directed line segment.

> **Definition:** Two directed line segments which have the same magnitude and the same direction are called **equivalent**.

This means that any non-zero vector $\vec{v}$ is equivalent to many other directed line segments – every directed line segment which is parallel to $\vec{v}$ in the same direction as $\vec{v}$ and is the same length as $\vec{v}$. For instance, the vector $\vec{v} = (2, 2)$, which goes up to the right with slope 1, and is $\sqrt{8}$ units long, is equivalent to the directed line segment from the point $P(-4, 0)$ to the point $Q(-2, 2)$, and is also equivalent to the directed line segment from the point $R(3, -1)$ to the point $S(5, 1)$, and to the directed line segment from the point $A(-3, 3)$ to the point $B(-1, 5)$, and so forth.



In some contexts, we want to replace a directed line segment by the vector which is equivalent to it, or replace a vector by an equivalent directed line segment which starts somewhere other than at the origin. We refer to this as translating the vector, or the directed line segment.

> **Definition:** The process of replacing a directed line segment $\overrightarrow{AB}$ with the equivalent vector is called **translating $\overrightarrow{AB}$ to the origin**. Similarly, the process of replacing the vector $\vec{v}$ with an equivalent directed line segment which starts at some point P is called **translating $\vec{v}$ to P**.

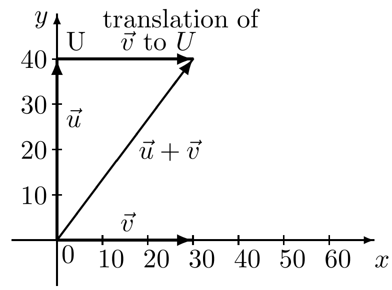
Notice that a vector or other directed line segment obtained by translation is always parallel to, and the same length as, the original directed line segment. This is because by definition, translating involves an equivalent directed line segment, i.e. one which has the same direction and the same length.

Addition of Vectors

We can add one vector to another, as long as they are in the same space, i.e both in $\Re^2$ or both in $\Re^3$. The way we do this is quite obvious. If I told you that $\vec{u}$ is the vector that travels 40 metres

North, and that $\vec{v}$ is the vector that travels 30 metres East, what would you suppose the vector $\vec{u}+\vec{v}$ would be? You'd probably guess that it's equivalent to going 40 metres North and then 30 metres East. And if you thought about it a bit more, you might realize that it should be the vector that goes directly from where you're starting to where you're ending up, instead of taking the less direct route. Because vectors don't turn corners. Each starts at the origin and goes in a straight line to its endpoint.

To add vector $\vec{v}$ to vector $\vec{u}$, we translate $\vec{v}$ to the endpoint of $\vec{u}$, which we can call U. This is like travelling 40 metres North, and *then* travelling 30 metres East. So the sum vector, $\vec{u}+\vec{v}$, is the vector that starts at the start of $\vec{u}$, which is the origin of course, and goes to the endpoint of the translation of $\vec{v}$ to U.



Adding two vectors when they're written in component form is even easier. All we need to do is to add the corresponding components.

Definition: For any vectors $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$ in $\Re^2$, the vector $\vec{u}+\vec{v}$, the **sum** of $\vec{u}$ and $\vec{v}$, is given by

$$\vec{u}+\vec{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$$

Similarly, for vectors in $\Re^3$, if $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$, the **sum** of $\vec{u}$ and $\vec{v}$ is

$$\vec{u}+\vec{v}=(u_1,u_2,u_3)+(v_1,v_2,v_3)=(u_1+v_1,u_2+v_2,u_3+v_3)$$

For instance, in the picture above, we have $\vec{u}=(0,40)$ (go 40 metres due North) and $\vec{v}=(30,0)$ (go 30 metres due east), and we get

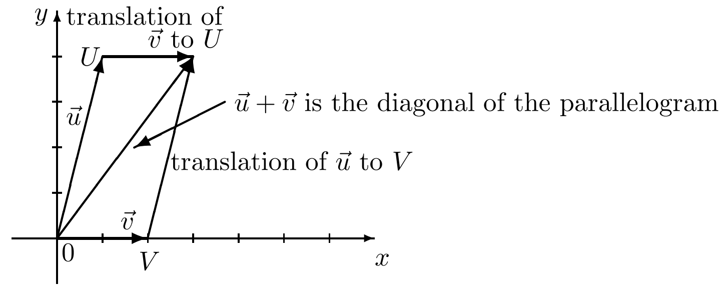
$$\vec{u}+\vec{v}=(0,40)+(30,0)=(0+30,40+0)=(30,40)$$

Example 1.11. Let $\vec{u}=(12,-4,7)$ and $\vec{v}=(-3,5,8)$. Find $\vec{u}+\vec{v}$.

Solution:

$$\vec{u}+\vec{v}=(12,-4,7)+(-3,5,8)=(12+(-3),(-4)+5,7+8)=(9,1,15)$$

There's another way to think about the sum of 2 vectors. We've seen that if we translate $\vec{v}$ to the endpoint of $\vec{u}$ (i.e. to U), the vector $\vec{u}+\vec{v}$ is the vector that goes from the start of $\vec{u}$ (i.e. the origin) to the endpoint of the translation of $\vec{v}$ to U. And of course the translation of $\vec{v}$ is parallel to the vector $\vec{v}$. If we also translate $\vec{u}$ to the endpoint of $\vec{v}$ (i.e. to V), then that will be a directed line segment which is parallel to $\vec{u}$. And those two sets of parallel lines form a parallelogram. That is, the two translations have the same endpoint. So the vector $\vec{u}+\vec{v}$ also goes from the start of $\vec{v}$ to the end of the translation of $\vec{u}$ to V. This vector is the diagonal of the parallelogram. (See diagram next page.)



Now let's think about something a bit different. What do you suppose we mean by the *negative* of a vector? For instance, if $\vec{u}$ is the vector which starts at the origin and goes 40 metres due North, what would the negative of this vector be? What direction do you suppose it goes? How long do you think it would be?

> **Definition:** The **negative** of a vector is the vector which is the same length, but has the opposite direction. We write the negative of $\vec{v}$ as $-\vec{v}$.

We know that a vector with the opposite direction to a vector $\vec{v}$ is collinear with $\vec{v}$ and hence is a scalar multiple of $\vec{v}$. For the direction to be opposite, the scalar must be negative. And for the length to be the same, the components of the vector musn't change in size, only in sign. That is, $-\vec{v}=(-1)\vec{v}$. So for instance for the vectors in Example 1.11 we get

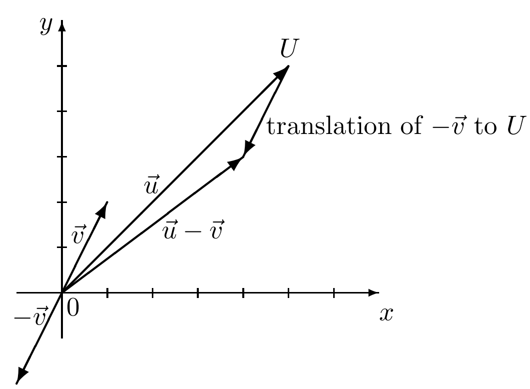
$$-\vec{u}=(-1)(12,-4,7)=((-1)12,(-1)(-4),(-1)(7))=(-12,4,-7)$$

and similarly $-\vec{v}=-(-3,5,8)=(3,-5,-8)$.

Theorem 1.3. *In component form, $-\vec{u}$ is the vector obtained by changing the sign of each component of $\vec{u}$. That is, $-\vec{u}=(-u_1,-u_2)$ in $\Re^2$, or $-\vec{u}=(-u_1,-u_2,-u_3)$ in $\Re^3$.*

Subtracting one vector from another

In general in mathematics, subtraction is the same as adding the negative. For instance, with numbers, we can think of $5-2$ as $5+(-2)$. And the same is true with subtraction of vectors. We define that $\vec{u}-\vec{v}$ means $\vec{u}+(-\vec{v})$. In terms of directed line segments, this means that we form the vector difference $\vec{u}-\vec{v}$ by adding $-\vec{v}$ to $\vec{u}$, i.e. by translating the vector $-\vec{v}$ to the endpoint of $\vec{u}$, i.e. to U. And of course $-\vec{v}$ is simply the vector with the same magnitude as $\vec{v}$ but in the opposite direction. And then the vector $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$ is the vector that goes from the origin, i.e. from the start of $\vec{u}$, directly to the endpoint of the translation of $-\vec{v}$ to U.



Of course, since subtracting $\vec{v}$ from $\vec{u}$ is the same as adding $-\vec{v}$ to $\vec{u}$, and since adding two vectors in component form simply involves adding corresponding components, then when we subtract one vector from another in component form, we just add the negative of each component, i.e., subtract corresponding components.

Definition: For any vectors $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ in $\Re^2$, the **difference** vector $\vec{u} - \vec{v}$ is given by

$$\vec{u} - \vec{v} = (u_1, u_2) - (v_1, v_2) = (u_1 - v_1, u_2 - v_2)$$

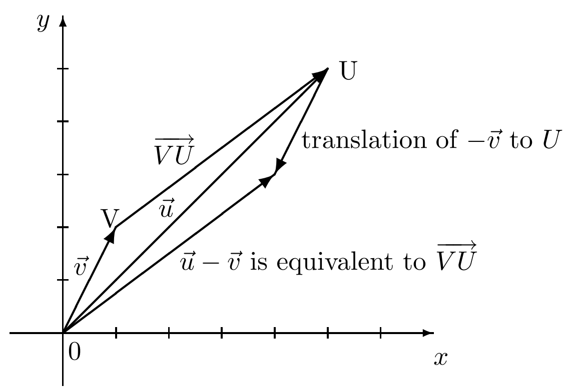
And for vectors in $\Re^3$, the **difference** vector $\vec{u} - \vec{v}$ is given by

$$\vec{u} - \vec{v} = (u_1, u_2, u_3) - (v_1, v_2, v_3) = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$

For instance, for $\vec{u} = (1, 2)$ and $\vec{v} = (3, 1)$ we get $\vec{u} - \vec{v} = (1, 2) - (3, 1) = (1 - 3, 2 - 1) = (-2, 1)$. Similarly, for $\vec{u} = (1, 0, 2)$ and $\vec{v} = (1, 1, -1)$ we have

$$\vec{u} - \vec{v} = (1, 0, 2) - (1, 1, -1) = (1 - 1, 0 - 1, 2 - (-1)) = (0, -1, 3)$$

Just as there was with addition of vectors, there's another way we can think of the subtraction of one vector from another. When we find a difference, i.e. one vector minus another, the resulting vector, when translated to the endpoint of the second vector mentioned, has as its endpoint the first vector mentioned. That is, if we translate $\vec{u} - \vec{v}$ to V, the endpoint of $\vec{v}$, we see that it goes to U, the endpoint of $\vec{u}$. Or we can think of it the other way around. If we draw the vectors $\vec{u}$ and $\vec{v}$ (each starting at the origin, of course), and draw the directed line segment $\overrightarrow{VU}$, from the endpoint of $\vec{v}$ to the endpoint of $\vec{u}$, and then translate $\overrightarrow{VU}$ to the origin, this translated vector **is** the vector $\vec{u} - \vec{v}$. We can see this in the diagram below.



Some Special Vectors

The unit vectors which run along the axes are very useful, and so they have special names. Consider $\Re^2$. The unit vector that runs along the positive x-axis, i.e. that runs from the origin for one unit in the positive horizontal direction (right), is called $\vec{\imath}$. And the unit vector that runs along the positive y-axis, i.e. that runs from the origin for one unit in the positive vertical direction (up), is called $\vec{\jmath}$. Similarly, in $\Re^3$ the unit vector along the positive x-axis is $\vec{\imath}$, the unit vector along the positive y-axis is $\vec{\jmath}$ and we also have the unit vector that runs along the positive z-axis, which is called $\vec{k}$.

Of course, the vector that starts at the origin, i.e. the point $(0, 0)$ or $(0, 0, 0)$, and runs for one unit along the positive part of one of the axes ends at the point which has 1 as the coordinate corresponding to the axis it moved along, and the other coordinate(s) is (are) still 0. For instance, the vector $\vec{\imath}$ in $\Re^2$ starts at the origin and ends at the point one unit to the right, which is the point (1,0).

Definition: The special unit vectors running along the positive axes in $\Re^2$ are:

$$\begin{aligned} \vec{\imath} &= (1,0) \\ \text{and } \vec{\jmath} &= (0,1) \end{aligned}$$

Similarly, the special unit vectors running along the positive axes in $\Re^3$ are:

$$\begin{aligned} \vec{\imath} &= (1,0,0) \\ \vec{\jmath} &= (0,1,0) \\ \text{and } \vec{k} &= (0,0,1) \end{aligned}$$

Any other vector $\vec{v}$ can be expressed in terms of these vectors. We multiply each of these special vectors by a scalar which is the corresponding component of $\vec{v}$ and then add them up. That is, we can express (v_1, v_2) as $v_1\vec{\imath} + v_2\vec{\jmath}$. Likewise, any vector (v_1, v_2, v_3) can be expressed as $v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$. So for instance $(2, -1) = 2\vec{\imath} - \vec{\jmath}$ and $(-3, 5, 17) = -3\vec{\imath} + 5\vec{\jmath} + 17\vec{k}$.

In this Unit we have learnt several vector operations: addition, subtraction and scalar multiplication. There are various properties of these operations which hold because of the way they are defined. (Some of them we've already seen; others we haven't talked about – but they're fairly obvious.) You should be aware of, and able to use, all of these properties, which are enumerated in the following Theorem.

Theorem 1.4. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be any vectors, all in $\Re^2$ or all in $\Re^3$. Let $\vec{0}$ be the zero vector in that same space. Let c and d be any scalars. Then the following properties hold:*

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
That is, addition of vectors is commutative.

(b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
That is, addition of vectors is associative.

(c) $\vec{u} + \vec{0} = \vec{u}$
That is, adding the zero vector to any vector leaves the vector unchanged.

(d) $\vec{u} + (-\vec{u}) = \vec{0}$
That is, the sum of a vector and its negative is the zero vector.

(e) $cd(\vec{u}) = c(d\vec{u})$
That is, to form the scalar multiple of a vector, where the scalar is a product of two scalars, it doesn't matter if the scalars are applied one at a time, or if the scalars are multiplied together before the vector is multiplied by them.

(f) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
That is, scalar multiplication of a vector is distributive over addition of scalars. So if the scalar by which a vector is to be multiplied is considered as the sum of two scalars, it doesn't matter if the vector is multiplied by each scalar separately, and then these new vectors added together, or if the two scalars are added together and then the vector is multiplied by that sum.

(g) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
That is, scalar multiplication of a vector is distributive over addition of vectors. So if the sum of two vectors is to be multiplied by a scalar, it doesn't matter whether the vectors are multiplied by the scalar separately, and then these new vectors added together, or whether the two vectors are added together and then the sum vector is multiplied by the scalar.

(h) $1\vec{u} = \vec{u}$
That is, multiplying any vector by the scalar 1 leaves the vector unchanged.

(i) $(-1)\vec{u} = -\vec{u}$
That is, the negative of a vector (i.e. the vector with the same magnitude but opposite in direction) is the vector obtained by multiplying the vector by the scalar -1.

(j) $0\vec{u} = \vec{0}$
That is, multiplying any vector by the scalar 0 produces the zero vector.