1A

1. **Short Problems.** Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Evaluate $\lim_{x \to -3} \frac{2x^2+5x-3}{x^2+x-6}$.

(b) Suppose that the demand curve for a certain commodity is modeled by:

$$p = \frac{3400q+1}{2q^2+q},$$

where p the price and q is the quantity. Find the limiting value of the revenue R as $q \to \infty$.

(c) How much money must initially be put in an account paying 8% annual interest, compounded continuously in order to have \$3000 at the end of 30 months? You may leave your answer in calculator-ready form.

(d) Find the slope of the graph of $y = f(g(x^2))$ at $x = 1$, where $f'(0) = 3$, $g(1) = 0$ and $g'(1) = -1$.

(e) Find the x-coordinates of all points where the tangent line to the graph of $y = \tan^{-1}(3x)$ is parallel to the line $3x - 2y + 7 = 0$. Note that $\tan^{-1}$ refers to the inverse tangent function, which is also denoted by arctan.

(f) Suppose the weight of an animal at time t is given by $\omega = \frac{216}{t+5} + t^{3/2}$ for $t \geq 0$. Use a derivative to determine whether the animal is gaining or losing weight when $t = 4$.

(g) A company produces 500 radios per day. At this production level, the marginal daily revenue is \$2.50 per radio. Estimate the change in the daily revenue if the company increases production by 20 radios per day.

(h) If a function $y = f(x)$ is differentiable at $x = 2$ and $f'(2) = 6$, find the limit $\lim_{h \to 0} \frac{3h}{f(2) - f(2+h)}$.

(i) If the derivative of f(x) is given by $f'(x) = \frac{2x}{1+x^2}$, find the interval or intervals on which $f(x)$ is concave up.

(j) Find the values of the constants a and b such that the function $f(x) = x^3+ax^2+bx+c$ has critical points at $x = 0$ and $x = 1$.

(k) Find the point in the first quadrant ($x > 0, y > 0$) on the hyperbola $x^2 - y^2 = 4$ which is closest to the point (6,0). You do not need to justify that your answer provides the minimum.

(l) Suppose the demand function for a certain product is given by $q = 2000e^{-kp}$, where $k > 0$ is a constant, p is the unit price and q is the quantity sold. Determine the value of k if the maximum revenue occurs at $p = 200$.

2A

(m) Let $c_0 + c_1x + c_2x^2 + \ldots$ be the Taylor series of the function $f(x) = \ln(2 + \cos x)$ at $a = 0$. Determine the value of c_2.

(n) Evaluate infinite series $\sum_{k=2}^{\infty} \frac{3^k}{k!}$. You may leave your answer in calculator-ready form.

Long Problems. In questions 2-6, show your work. No credit will be given without the correct accompanying work.

2. Let $f(x) = 2 + x^2$ and $g(x) = -1 + x^3$.
   (a) Graph $f(x)$ and $g(x)$ on the same set of axes over the interval $-2 \leq x \leq 2$ in such a way that it is clear that the two curves intersect in exactly one point with an x-coordinate a satisfying $1 < a < 2$.
   (b) Make an initial guess $x_0 = 2$ for the value of a and then use the Newton-Raphson Method to determine the next approximation x_1. Note that the Newton-Raphson Method is also called the Newton Method.

3. Let $y = f(x)$ be a function defined for $-\infty < x < \infty$ such that

$$f(0) = 1, \lim_{x\to\infty} f(x) = 1, f'(x) = (2x - x^2)e^{-x}.$$

   (a) Find the interval or intervals on which $f(x)$ is decreasing.
   (b) Find the interval or intervals on which $f(x)$ is concave down.
   (c) Sketch the graph of $y = f(x)$ indicating where all local maxima, local minima and inflection points occur. (Hint: $\sqrt{2} \approx 1.4$.)

4. A right cylindrical can is to be constructed to hold $375\pi \text{m}^3$ of oil. The cost of the material used for the top and bottom of the can is \$3/m and the cost of the material used for the curved side is \$2/m. Find the radius of the can that will minimize the cost of the material used to construct the can. You do not need to justify that your answer provides the minimum.

5. A point is moving along the x-axis in the positive direction at a constant rate of 5 units per second.
   (a) Where is the point when its distance from the point (0,1) in the xy-plane is increasing at a rate of 4 units per second?
   (b) Where is the point when its distance from the point (0,1) in the xy-plane is increasing at a rate of 6 units per second? Explain.

6. At the present instant the consumer price index (CPI) in a country is 120, and is increasing at a rate of 10 points per year. Also, at the present instant this rate of increase is itself increasing at a rate of 4 points per year^2. Use calculus and all of this information to estimate the CPI in the country 6 months from now.

1. **Short Problems.** Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

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Solution:

$$\lim_{x\to -3} \frac{2x^2+5x-3}{x^2+x-6} \overset{\boxed{1p}}{=} \lim_{x\to -3} \frac{(2x-1)(x+3)}{(x-2)(x+3)} =$$

$$= \lim_{x\to -3} \frac{2x-1}{x-2} \overset{\boxed{1p}}{=} \frac{2(-3)-1}{(-3)-2} = \frac{7}{5} \quad \boxed{1p} \ .$$

(b) Suppose that the demand curve for a certain commodity is modeled by:

$$p = \frac{3400q+1}{2q^2+q},$$

where p the price and q is the quantity. Find the limiting value of the revenue R as $q \to \infty$.

Solution:

$$\lim_{q\to\infty} R = \lim_{q\to\infty} (pq) = \lim_{q\to\infty} \frac{3400q^2+q}{2q^2+q} \overset{\boxed{1p}}{=} \frac{3400}{2} = 1700 \quad \boxed{2p} \ .$$

(c) How much money must initially be put in an account paying 8% annual interest, compounded continuously in order to have \$3000 at the end of 30 months? You may leave your answer in calculator-ready form.

Solution:$3000 = Pe^{(0.08)(2.5)}$ $\boxed{1p}$ (30 months means 2.5 years). Here P is the initial amount. Thus, $P = 3000e^{(-0.08)(2.5)} = 3000e^{-0.2}$ $\boxed{2p}$.

(d) Find the slope of the graph of $y = f(g(x^2))$ at $x = 1$, where $f'(0) = 3$, $g(1) = 0$ and $g'(1) = -1$.

Solution:$y' = f'(g(x^2)) \cdot g'(x^2) \cdot 2x$ $\boxed{1p}$. Thus,

$$y'(1) = f'(g(1)) \cdot g'(1) \cdot 2 = f'(0) \cdot (-1) \cdot 2 = 3 \cdot (-1) \cdot 2 = -6 \quad \boxed{2p} \ .$$

(e) Find the x-coordinates of all points where the tangent line to the graph of $y = \tan^{-1}(3x)$ is parallel to the line $3x - 2y + 7 = 0$. Note that $\tan^{-1}$ refers to the inverse tangent function, which is also denoted by arctan.

Solution: $y' = \dfrac{3}{1+(3x)^2}$ [1p]. Need to find x such that $y' = \dfrac{3}{2}$ [1p] ($\dfrac{3}{2}$ is the slope of the line $3x - 2y + 7 = 0$). Solve the equation $\dfrac{3}{1+(3x)^2} = \dfrac{3}{2}$ to get $x = \pm\dfrac{1}{3}$ [1p].

(f) Suppose the weight of an animal at time t is given by $\omega = \dfrac{216}{t+5} + t^{3/2}$ for $t \geq 0$. Use a derivative to determine whether the animal is gaining or losing weight when $t = 4$.

Solution:

$$\frac{d\omega}{dt} = 216\left(-\frac{1}{2}\right)(t+5)^{-3/2} + \frac{3}{2}t^{1/2} \quad \boxed{1p}.$$

At $t = 4$:

$$\frac{d\omega}{dt} = -108 \cdot (4+5)^{-3/2} + \frac{3}{2} \cdot 4^{1/2} = -\frac{108}{27} + 3 = -4 + 3 = -1 < 0 \quad \boxed{1p}.$$

Thus, the animal is **loosing weight** [1p].

(g) A company produces 500 radios per day. At this production level, the marginal daily revenue is \$2.50 per radio. Estimate the change in the daily revenue if the company increases production by 20 radios per day.

Solution:

$$R(520) - R(500) \approx R'(500)(520 - 500) \overset{\boxed{1p}}{=} 2.5 \cdot 20 = 50 \quad \boxed{2p}.$$

(h) If a function $y = f(x)$ is differentiable at $x = 2$ and $f'(2) = 6$, find the limit $\lim\limits_{h \to 0} \dfrac{3h}{f(2) - f(2+h)}$.

Solution:

$$\lim_{h \to 0} \frac{3h}{f(2) - f(2+h)} = \lim_{h \to 0} \frac{-3}{\dfrac{f(2+h) - f(2)}{h}} \overset{\boxed{1p}}{=} \frac{-3}{f'(2)} = \frac{-3}{6} = -\frac{1}{2} \quad \boxed{2p}.$$

(i) If the derivative of f(x) is given by $f'(x) = \dfrac{2x}{1+x^2}$, find the interval or intervals on which $f(x)$ is concave up.

Solution: $f'' = \dfrac{2 \cdot (1+x^2) - 2x \cdot 2x}{(1+x^2)^2} \overset{\boxed{1p}}{=} \dfrac{2(1-x^2)}{(1+x^2)^2}$.

$f'' = 0$ for $x = \pm 1$ [1p].

$f'' < 0$ for $x < -1$, $f'' > 0$ for $-1 < x < 1$ and $f'' < 0$ for $x > 1$ (using some test points - for example -2, 0, 2). Thus, f'' is concave up for $-1 < x < 1$ [1p].

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(j) Find the values of the constants a and b such that the function $f(x) = x^3+ax^2+bx+c$ has critical points at $x = 0$ and $x = 1$.

Solution: $f(x) = 3x^2 + 2ax + b$ $\boxed{1\text{p}}$. I have $0 = f'(0) = b$ and $0 = f'(1) = 3 + 2a + b$. Thus, $a = -\dfrac{3}{2}$ $\boxed{1\text{p}}$ and $b = 0$ $\boxed{1\text{p}}$.

(k) Find the point in the first quadrant ($x > 0, y > 0$) on the hyperbola $x^2 - y^2 = 4$ which is closest to the point (6,0). You do not need to justify that your answer provides the minimum.

Solution: The distance d satisfies:

$$d = \sqrt{(x-6)^2 + y^2} = \sqrt{(x-6)^2 + x^2 - 4} \overset{\boxed{1\text{p}}}{=} \sqrt{2x^2 - 12x + 32}.$$

Now:

$$\frac{dd}{dt} = \frac{1}{2\sqrt{2x^2 - 12x + 32}} \cdot (4x - 12).$$

Thus $\dfrac{dd}{dt} = 0$ for $4x - 12 = 0$ or $x = 3$. From $x^2 - y^2 = 4$ (the point is on hyperbola), $y = \pm\sqrt{5}$ $\boxed{1\text{p}}$. Since $y > 0$, $(3, \sqrt{5})$ is the solution $\boxed{1\text{p}}$.

(l) Suppose the demand function for a certain product is given by $q = 2000e^{-kp}$, where $k > 0$ is a constant, p is the unit price and q is the quantity sold. Determine the value of k if the maximum revenue occurs at $p = 200$.

Solution: $R = pq = 200pe^{-kp}$ $\boxed{1\text{p}}$ and $\dfrac{dR}{dp} = 2000e^{-kp} + 2000pe^{-kp} \cdot (-k) = 2000e^{-kp}(1 - kp)$. For $p = 200$ I have: $\dfrac{dR}{dp} = 0$, which gives $1 - 200k = 0$, or $k = \dfrac{1}{200}$ $\boxed{2\text{p}}$.

(m) Let $c_0 + c_1x + c_2x^2 + \ldots$ be the Taylor series of the function $f(x) = \ln(2 + \cos x)$ at $a = 0$. Determine the value of c_2.

Solution: $f' = \dfrac{1}{2 + \cos x} \cdot (-\sin x) = \dfrac{-\sin x}{2 + \cos x}$ $\boxed{1\text{p}}$.

$$f'' = \frac{(-\cos x)(2 + \cos x) - (-\sin x)(-\sin x)}{(2 + \cos x)^2}.$$

$$f''(0) = \frac{(-1)(2 + 1) - (-0)(-0)}{(2 + 1)^2} = -\frac{1}{3} \quad \boxed{1\text{p}}.$$

Thus, $c_2 = \dfrac{f''(0)}{2!} = -\dfrac{1}{6}$ $\boxed{1\text{p}}$.

(n) Evaluate infinite series $\displaystyle\sum_{k=2}^{\infty} \frac{3^k}{k!}$. You may leave your answer in calculator-ready form.

Solution:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + \frac{x}{1!} + \sum_{k=2}^{\infty} \frac{x^k}{k!}.$$

For $x = 3$ I have: $e^3 = 1 + \frac{3^1}{1!} + \sum_{k=2}^{\infty} \frac{3^k}{k!}$ [1p] . Thus, $\sum_{k=2}^{\infty} \frac{3^k}{k!} = e^3 - 1 - \frac{3^1}{1!} = e^3 - 4$ [2p] .

Long Problems. In questions 2-6, show your work. No credit will be given without the correct accompanying work.

2. Let $f(x) = 2 + x^2$ and $g(x) = -1 + x^3$.
   (a) Graph $f(x)$ and $g(x)$ on the same set of axes over the interval $-2 \le x \le 2$ in such a way that it is clear that the two curves intersect in exactly one point with an x-coordinate a satisfying $1 < a < 2$.
   (b) Make an initial guess $x_0 = 2$ for the value of a and then use the Newton-Raphson Method to determine the next approximation x_1. Note that the Newton-Raphson Method is also called the Newton Method.

Solution: See plot on the last page. Of interest are the points (1,0) and (2,7) (on the graph of g) and (1,3) and (2,6) (on the graph of f) [3p] .

Let $h(x) = f(x) - g(x) = (2 + x^2) - (-1 + x^3) = 3 + x^2 - x^3$ [2p] .

Need to approximate a which satisfies $h(a) = 0$.

Use $x_0 = 2$. Thus, $h(2) = 3 + 2^2 - 2^3 = -1$ [1p] . Furthermore, $h'(x) = 2x - 3x^3$ which gives $h'(2) = 2 \cdot 2 - 3 \cdot 2^2 = -8$ [1p] .

I have:

$$x_1 \overset{[2p]}{=} x_0 - \frac{h(x_0)}{h'(x_0)} = 2 - \frac{h(2)}{h'(2)} = 2 - \frac{-1}{-8} = 2 - \frac{1}{8}$$ [3p] .

3. Let $y = f(x)$ be a function defined for $-\infty < x < \infty$ such that

$$f(0) = 1, \lim_{x\to\infty} f(x) = 1, f'(x) = (2x - x^2)e^{-x}.$$

   (a) Find the interval or intervals on which $f(x)$ is decreasing.
   (b) Find the interval or intervals on which $f(x)$ is concave down.
   (c) Sketch the graph of $y = f(x)$ indicating where all local maxima, local minima and inflection points occur. (Hint: $\sqrt{2} \approx 1.4$.)

Solution:$f'(x) = 0$ for $x = 0$ and $x = 2$ [2p] - these are the critical points. Use test points (-1, 1, 3) to see that $2x - x^2$ (implicitly $(2x - x^2)e^{-x}$, since $e^{-x} > 0$) changes

7A

sign in 0 from negative to positive and from positive to negative in 2. Thus, $f(x)$ is increasing on (0,2) and decreasing on $(-\infty, 0)$ and $(2, \infty)$ $\boxed{2p}$.

$$f''(x) = (2-2x)e^{-x} + (2x - x^2)e^{-x} \cdot (-1) = \\ = e^{-x}[2 - 2x + (2x - x^2)(-1)] = e^{-x}(x^2 - 4x + 2) \quad \boxed{2p} .$$

$f''(x) = 0$ for $x^2 - 4x + 2 = 0$. This gives $x = 2 \pm \sqrt{2}$ $\boxed{2p}$.

Using some test points (0, 2, 4 for example) one can get the sign of f''.

I have $f'' > 0$ on $(-\infty, 2-\sqrt{2})$, $f'' < 0$ on $(2-\sqrt{2}, 2+\sqrt{2})$ and $f'' > 0$ on $(2+\sqrt{2}, \infty)$.

Thus $f(x)$ is concave down on $(2-\sqrt{2}, 2+\sqrt{2})$ $\boxed{1p}$.

The graph is provided on the last page $\boxed{3p}$.

$(0, f(0))$ is a local minimum and $(2, f(2))$ is a local maximum $\boxed{1p}$.

$(2-\sqrt{2}, f(2-\sqrt{2}))$ and $(2+\sqrt{2}, f(2+\sqrt{2}))$ are the inflection points $\boxed{1p}$.

4. A right cylindrical can is to be constructed to hold $375\pi m^3$ of oil. The cost of the material used for the top and bottom of the can is \$3/m and the cost of the material used for the curved side is \$2/m. Find the radius of the can that will minimize the cost of the material used to construct the can. You do not need to justify that your answer provides the minimum.

Solution: Let r be the radius of the base circle and h the height of the cylinder. The cost is $C = 3 \cdot \pi r^2 + 3 \cdot \pi r^2 + 2 \cdot 2\pi r h = \pi(6r^2 + 4rh)$ $\boxed{4p}$ (adding up the costs for bottom, top and lateral area, respectively).

Want to minimize the cost C.

Additionally, the volume $V = \pi r^2 h$ satisfies $V = 375\pi$. This gives $\pi r^2 h = 375\pi$ or $h = \frac{375}{r^2}$.

Thus,

$$C = \pi(6r^2 + 4rh) = \pi\left(6r^2 + 4r \cdot \frac{375}{r^2}\right) = \pi\left(6r^2 + \frac{1500}{r}\right) \quad \boxed{3p} .$$

Now $\frac{dC}{dr} = \pi\left(12r - \frac{1500}{r^2}\right)$ $\boxed{3p}$ and $\frac{dC}{dr} = 0$ for $12r - \frac{1500}{r^2} = 0$, or $12r = \frac{1500}{r^2}$, or $r = 5$ $\boxed{4p}$. This value of the radius provides the minimal cost.

5. A point is moving along the x-axis in the positive direction at a constant rate of 5 units per second.

(a) Where is the point when its distance from the point (0,1) in the xy-plane is increasing at a rate of 4 units per second?

(b) Where is the point when its distance from the point (0,1) in the xy-plane is increasing at a rate of 6 units per second? Explain.

The distance d between $(x, 0)$ and (0,1) satisfies:

$$d = \sqrt{(x-0)^2 + (0-1)^2} = \sqrt{x^2+1} = (x^2+1)^{1/2} \quad \boxed{1p}\ .$$

Differentiate with respect to time t and use $\frac{dx}{dt} = 5$.

$$\frac{dd}{dt} = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \cdot \frac{dx}{dt} = \frac{5x}{\sqrt{x^2+1}} \quad \boxed{2p}\ .$$

Need to solve for part (a) the equation $\frac{dd}{dt} = 4$, or $\frac{5x}{\sqrt{x^2+1}} = 4$. Square the last relation to get $\frac{25x^2}{x^2+1} = 16$. Cross-multiply: $25x^2 = 16(x^2+1)$. This yields $x^2 = \frac{16}{9}$, or $x = \pm\frac{4}{3}$ $\boxed{2p}$. Since $x > 0$, only $x = \frac{4}{3}$ works and the point is $\left(\frac{4}{3}, 0\right)$ $\boxed{1p}$.

Need to solve for part (b) the equation $\frac{dd}{dt} = 6$, or $\frac{5x}{\sqrt{x^2+1}} = 6$. Square the last relation to get $\frac{25x^2}{x^2+1} = 36$. Cross-multiply: $25x^2 = 36(x^2+1)$. This yields $x^2 = -\frac{36}{11} < 0$ and there is no such real number x $\boxed{3p}$. Therefore, there is no point where the distance d is increasing at a rate of 6 units per second $\boxed{3p}$.

6. At the present instant the consumer price index (CPI) in a country is 120, and is increasing at a rate of 10 points per year. Also, at the present instant this rate of increase is itself increasing at a rate of 4 points per year^2. Use calculus and all of this information to estimate the CPI in the country 6 months from now.

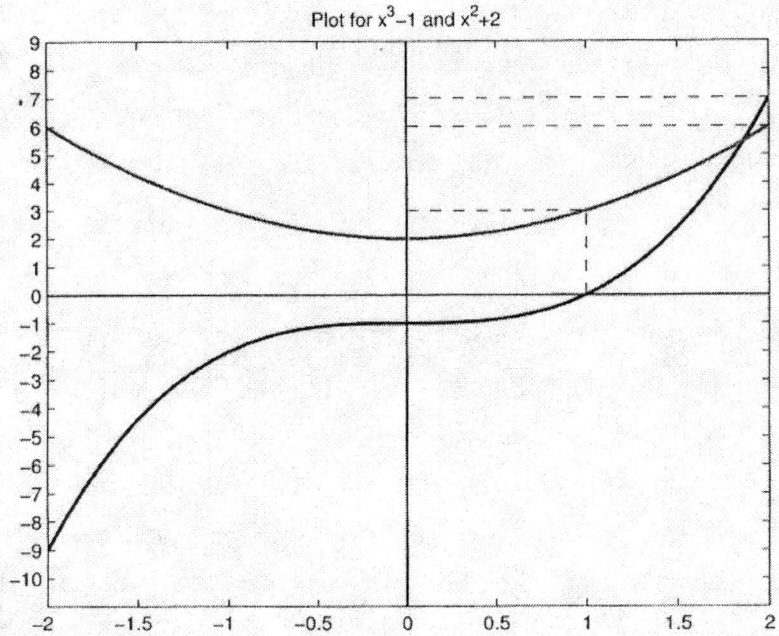
Solution:I have $CPI(0) = 120$, $CPI'(0) = 10$, $CPI''(0) = 4$. Need to estimate $CPI(0.5)$. Use the Taylor Polynomial of second order around $a = 0$ for CPI:

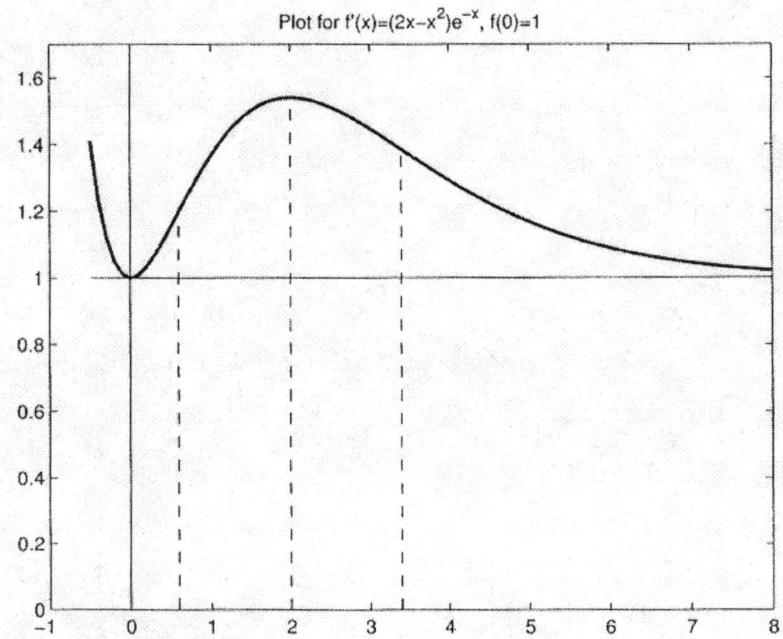
$$P_2(x) = CPI(0) + \frac{CPI'(0)}{1!}(x-0)^1 + \frac{CPI''(0)}{2!}(x-0)^2 \quad \boxed{1p}\ .$$

Plug $x = 0.5$ to get:

$$CPI(0.5) \approx P_2(0.5) = 120 + \frac{10}{1!}(0.5-0)^1 + \frac{4}{2!}(0.5-0)^2 = 125.5 \quad \boxed{4p}\ .$$

9A


Plot for x^3−1 and x^2+2
9
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10
−2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Plot for f'(x)=(2x−x^2)e^−x, f(0)=1
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−1
0
1
2
3
4
5
6
7
8