University of Ottawa

Department of Mathematics and Statistics

MAT1300E - Oct. 9, 2019 - Midterm Test 1

Professor: Aziz Khanchi

Family Name ____________________ First Name ____________________

Student # ____________________ DGD # ____________________

- Time: 17:30–18:45.
- **NO Calculators. NO Notes. NO Books.**
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- **Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.**

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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Question	1	2	3	4	5	6	7	8	Total
Answer					X	X	X	X	X
Mark									
Maximum	2	2	2	2	4	4	5	4	25

Part I: Four multiple choice questions, no partial marks.

Question 1. Solve the equation

$$\log x + \log(x-3) = 1.$$

(A) $x = 5, x = -2$
(B) $x = 5$
(C) $x = -2$
(D) $x = -5, x = 2$
(E) $x = -5, x = -2$
(F) $x = -5$

Solution:

$$\log x + \log(x-3) = \log x(x-3) = 1 \Rightarrow x(x-3) = 10 \Rightarrow x^2 - 3x - 10 = 0$$
$$\Rightarrow (x-5)(x+2) = 0 \Rightarrow x = 5 \text{ or } x = -2$$

Notice that -2 is not a valid value for x in the original equation, since the logarithm of a negative number is undefined.

Question 2. Find all values of x where the following piecewise function is discontinuous.

$$f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \le x \le 3 \\ 5 - x & \text{if } x > 3 \end{cases}$$

(A) $x = 1, x = 3$
(B) $x = 1$
(C) $x = 3$
(D) $x = 1, x = 0$
(E) $x = 1, x = 0, x = 3$
(F) $x = -1, x = -3$

Solution: Since each piece of the function is a polynomial, the only x-values where f might be discontinuous are 1 and 3. We investigate 1, first.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + 1 = 1 + 1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 - 3x + 4 = 1^2 - 3 + 4 = 2$$

Furthermore, $f(1) = 1^2 - 3 + 4 = 2$, so $\lim_{x \to 1} f(x) = f(1) = 2$. Thus f is continuous at $x = 1$.

Now, let's investigate $x = 3$. From the left,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - 3x + 4 = 3^2 - 3(3) + 4 = 4$$

From the right,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5 - x = 5 - 3 = 2$$

Therefore, $\lim_{x \to 3} f(x)$ doesn't exist at $x = 3$ and f is not continuous at $x = 3$.

Question 3. Find the equation of the tangent line to the graph of $g(x) = e^{-5x}$ which is parallel to $y = -5x$.

(A) $y = -5x - 1$
(B) $y = -5x$
(C) $y = 5x + 1$
(D) $y = -5x + 1$
(E) $y = -5x + 2$
(F) $y = 5x + 2$

Solution: Using the chain rule, $g'(x) = -5e^{-5x}$, so $g'(0) = -5e^0 = -5$. Also, $g(0) = \frac{1}{e^0} = 1$. Therefore, the equation of the tangent line is

$$y - 1 = -5(x - 0)$$

or

$$y = -5x + 1.$$

Question 4. The demand for a product at the price of x dollars is given by $D(x) = x \ln x$. What is the marginal revenue when the price is 100 dollars?

(A) $100 \ln 100$
(B) $100(2 \ln 100 + 1)$
(C) $1/100$
(D) $\ln 100 + 1$
(E) $100(2 \ln 100 + 2)$
(F) 1

Solution: Revenue is the product of demand and price: $R(x) = x^2 \ln x$. Therefore,

$$R'(x) = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x$$

$$R'(100) = 200 \ln 100 + 100 = 100(2 \ln 100 + 1)$$

Part II: Long Answer Questions, you have to show your work clearly.

Question 5. Find the vertical and horizontal asymptote(s) of the graph of the function

$$f(x) = \frac{x^2+1}{6x^2-x-1}$$

Vertical asymptote(s) ____________________

Horizontal asymptote(s) ____________________

Show your work:

Solution: For vertical asymptotes, first find the roots of the denominator.

$$6x^2 - x - 1 = 0 \Rightarrow 6(x - \frac{1}{2})(x + \frac{1}{3}) = 0 \Rightarrow x = \frac{1}{2}, x = -\frac{1}{3}$$

Note that numerator is not zero at these two points. Therefore, $x = \frac{1}{2}$ and $x = -\frac{1}{3}$ are two vertical asymptotes.

For horizontal asymptotes, we need to evaluate f as $x \to \infty$ and $x \to -\infty$. Note that for large positive or negative x values

$$f(x) = \frac{x^2+1}{6x^2-x-1} \approx \frac{x^2}{6x^2} = \frac{1}{6}$$

Therefore, horizontal asymptote of the function is $y = \frac{1}{6}$.

Question 6. Find the following limit

$$\lim_{x\to 2^-} \frac{x^2+x-6}{|x-2|}.$$

Solution:

$$\lim_{x\to 2^-} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 2^-} \frac{x^2+x-6}{-(x-2)} = \lim_{x\to 2^-} \frac{(x+3)(x-2)}{-(x-2)} = \lim_{x\to 2^-} -(x+3) = -(2+3) = -5$$

Question 7. Use the definition of derivative as a limit to determine $f'(x)$ where

$$f(x) = \frac{4}{x}.$$

Solution:

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{4x-4(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{4x-4x-4h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-4h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-4}{x(x+h)} \\
&= \lim_{h\to 0} \frac{-4}{x^2+xh} = \frac{-4}{x^2+0} = \frac{-4}{x^2}
\end{aligned}$$

Question 8. *For this question, you don't need to calculate or simplify your answer.*

The island of Manhattan was sold for \$24 in 1626. Suppose the money had been invested in an account with an interest rate of 5% per year compounded monthly.

- Find a formula for $P(t)$, the value of the account in year t. How much money would be in the account in the year 2019 (after 393 years)?

  Solution: $P(t) = P_0(1 + \frac{r}{12})^{12t}$ where t is the number of years.
  $P(393) = 24(1 + \frac{0.05}{12})^{12(393)}$

- If the 5% interest was compounded continuously, how many years would it take to double the initial money?

  Solution: In this case, $P(t) = P_0 e^{rt} = 24e^{0.05t}$
  Find t such that $48 = 24e^{0.05t} \Rightarrow 2 = e^{0.05t} \Rightarrow \ln 2 = 0.05t \Rightarrow t = \frac{\ln 2}{0.05}$