MAT 1330: Calculus for Life Sciences:
Course Guide 2019

The course is based on the book
Calculus for the Life Sciences: Modeling the Dynamics of Life
by F.R. Adler and M. Lovric.
These notes were developed at the University of Ottawa for MAT1330,
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including themes from the Self-Regulated Learning Module developed by Alison Flynn.

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Preliminaries and expectations

- To be good at math (in fact at anything): practice, and know yourself!
- Find your resources.
- Do 5 problems every day. If you miss a day, do 10 the next.
- Check with your friends, your TA, the math help centre, and with your professor.
- At the University level, you are expected to be self-directed in your learning. Now, you may be tested on skills you were *told* to cover on your own (that may not have been explicitly covered during lectures).
- Learning math is learning a language, and it's a powerful one, as you'll see over time. You need to build vocabulary and learn the grammar. Correct usage and interpretation of notation and terminology is crucial!
- This document provides a short summary for each class with reference to the corresponding book chapter and practice problems. The problems in these notes are taken from previous MAT1330 assignments and exams. Use it well.

Contents

1 Review of the basics

GOAL: Recall and practice the high-school material that you will need for this course.

SKILLS:

Algebraic manipulations I: powers, roots, exponentials, logarithms

- simplify $\dfrac{\sqrt{x^{1/2}y^5}}{x^5y^{1/4}}$, where $x, y, z > 0$
- solve $2^{x+3} = 16^{2x-1}$ and $2^{2x+3} = 3^{4x-1}$
- solve $\log_{10}(x+5) - \log_{10}(x-1) = \log_{10}(x+1)$ and $2\ln(x) - \ln(x+4) = \ln(2)$

Algebraic manipulations II: simplify multiple fractions, rationalize a denominator

- simplify $\dfrac{4+\frac{1}{k}}{\frac{5}{k}-2}$
- rationalize the denominator $\frac{1}{\sqrt{10}-3}$

Polynomials: solve quadratic equations, factor a polynomial, long division

- solve $\dfrac{1}{x} + \dfrac{1}{x^2} = 1$
- solve $m = \sqrt{m+6}$
- solve $\dfrac{4x}{1+x} = 3x$
- factor $x^3 + 1000$
- divide $x^3 + x^2 + \dfrac{5}{4}x + 3$ by $x + \dfrac{3}{2}$

Inequalities: handle and solve inequalities, absolute values

- easy: $\dfrac{x}{2} - 3 > 5$
- harder $\dfrac{2}{x} - 3 > 5$
- solve $|x^2 - 5| = 1$
- solve $|\frac{x}{2} - 3| > 5$

Check out the history and applications of the quadratic equation (coming out of a debate in the British House of Commons on the subject of quadratic equations):

https://plus.maths.org/content/os/issue29/features/quadratic/index

1.1 Practice makes progress

A good portion of this material is covered in the textbook for this course, namely in sections 1.3 and 1.4 (second edition) [0.2 and 0.3 (first edition)]. Selected exercises from the book are:

- second edition: 1.3: 3-8, 13-35; 1.4: 9-20, 33-38, 50-59
- first edition: 0.2: 3-8, 13-35; 0.3: 13-20, 33-38, 50-59

The following questions are taken from old exams for this course.

Question 1: Simplify the following expressions.

$$\text{(a)}\ \frac{\left(x^{1/2}y^{1/3}\right)^{-1/2}}{x^2y^3} \qquad \text{(b)}\ \frac{\left(x^{1/2}y^{1/3}\right)^{-1/2}}{x^3y^2} \qquad \text{(c)}\ \left(\frac{x^{3/4}y^3}{x^{-1/4}y}\right)^2\left(\frac{xy}{\sqrt[3]{y}}\right)$$

Question 2: Rationalize the denominator.

$$\text{(a)}\ \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}+\sqrt{5}} \qquad \text{(b)}\ \frac{\sqrt{4}+\sqrt{x}}{\sqrt{4}-\sqrt{x}} \qquad \text{(c)}\ \frac{\sqrt{3}-2}{\sqrt{3}+2}$$

Question 3: Simplify the following expressions.

$$\text{(a)}\ \frac{1-x^3}{1-x} \qquad \text{(b)}\ \frac{1+x^3}{1+x} \qquad \text{(c)}\ \frac{8-x^3}{x-2}$$

Question 4: Simplify the following expressions.

$$\text{(a)}\ \frac{1}{\frac{1}{x}+\frac{1}{x-1}} \qquad \text{(b)}\ \frac{a^{-1}+x^{-2}}{a^{-2}-x^{-1}} \qquad \text{(c)}\ \frac{1}{\frac{3}{x}-\frac{1}{x+3}}$$

Question 5: Find all solutions of the following equations.

$$\text{(a)}\ \ln(x+3)+\ln(x+4)=\ln(6) \qquad \text{(b)}\ \log_{10}(x+3)+\log_{10}(x+5)=\log_{10}(3)$$
$$\text{(c)}\ \log_2(x+3)+\log_2(x+4)=\log_2(2) \qquad \text{(d)}\ \ln(x+1)+\ln(x-4)=\ln(2x+10)$$

Question 6: Find all values of k for which the equation has only one solution.

$$\text{(a)}\ x^2+2kx+9k-8=0 \qquad \text{(b)}\ x^2+2kx+3k+40=0 \qquad \text{(c)}\ x^2+2kx+11k-28=0$$

Question 7: Find all solutions of the following equations.

$$\text{(a)}\ 3x+\frac{4x}{x+1}=5x \qquad \text{(b)}\ 3x+\frac{5x}{x+2}=5x \qquad \text{(c)}\ 3x+\frac{7x}{x+2}=5x$$

Question 8: Find all x for which the following inequality is true.

(a) $\frac{1}{x-3} > \frac{1}{x+2}$ (b) $\frac{1}{x-4} < \frac{1}{x+3}$ (c) $\frac{1}{x-5} < \frac{1}{x+1}$

Question 9: Solve for x in terms of y and z.

(a) $\frac{1}{x} + \frac{2}{y} = \frac{7}{z}$ (b) $\frac{1}{y} - \frac{2}{x} = \frac{5}{z}$ (c) $\frac{1}{y} - \frac{2}{x} = \frac{5}{z}$

Question 10: Find all solutions of the following equation.

(a) $|x^3 - 5| = 5$ (b) $|x^3 - 4| = 4$ (c) $|2 - x^3| = 2$

Question 11: Solve the following equations.

1. Find the value of x so that $\log_{10}(x^4 y^3) = 18$ when $\log_{10}(y) = 2$.
2. Find the value of x so that $\log_5(x^3 y^5) = 9$ when $\log_5(y) = 3$.
3. Find the value of x so that $\ln(x^3 y^5) = 21$ when $\ln(y) = 3$.

Question 12: Which of the following is **not true** for positive real numbers a, b? (Here, we abbreviate $\log = \log_{10}$.)

A. $\log(a^x b^y) = x \log a + y \log b$;

B. $\left(\frac{a}{b}\right)^x = a^x b^{-x}$;

C. $a^b = 10^{b \log a}$;

D. $\log(a^x + b^x) = x \log(a + b)$;

E. $\sqrt{a^x} = a^{x/2}$.

2 Review of functions

GOAL: Recall and practice the high-school material that you will need for this course.

- **Notation:** $y = f(x)$, domain, range, assignment, name
- **Polynomial functions**
  - linear: $y = f(x) = mx + b$
  - quadratic: $y = f(x) = ax^2 + bx + c$ (find zeros!)
  - higher order: find zeros, long division
- **Potential characteristics of functions**
  - symmetry: even $f(x) = f(-x)$ or odd $f(x) = -f(-x)$ functions
  - transformations: $y = f(ax + b) + c$
  - composition of functions: $(f \circ g)(x) = f(g(x))$
- **Rational functions:** fractions of polynomials
  - domain of definition, e.g. $y = f(x) = \frac{1}{x+1}$
- **Functions and inequalities**
  - comparing graphs: is $f(x) > g(x)$?, e.g. $f(x) = 3x - 2$ and $g(x) = x^2$
  - root functions and domains, e.g. $y = f(x) = \sqrt{\frac{2}{x} - 8}$
  - rational functions, e.g. $\frac{4-x}{2x+1} > 5$
- **Absolute value function**
  - definition: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.
  - for the graph: flip the part of the graph that is below the x-axis above it
- **Trigonometric functions** and identities
  - $\sin(x)$, $\cos(x)$, $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$
  - $\sin^2(x) + \cos^2(x) = 1$
- **Exponential and logarithm functions**
  - $y = f(x) = e^x = \exp(x)$, $y = e^{-x} = \frac{1}{e^x}$, range?
  - inverse function: $y = \ln(x)$ if $e^y = x$, domain of definition?

2.1 Practice makes progress

A good portion of this material is covered in the textbook for this course, namely in sections 1.3 and 1.4 (second edition) [0.2 and 0.3 (first edition)]. Several nice and instructive examples for how to use functions in simple models are presented in sections 2.1, 2.2, and 2.3 (second edition) [1.1, 1.2, and 1.3 (first edition)]. Selected exercises from the book are:

- second edition: **2.2**: 23–26, 31–41; **2.3**: 64–71
- first edition: **1.2**: 23–26, 31–41; **1.3**: 52–59

Question 1: Find $f(a+3)$ when the function f is given by

$$\text{(a)}\ \ f(x) = \frac{x^2+1}{x-1}, \qquad \text{(b)}\ \ f(x) = \frac{x^2-2}{x+1}, \qquad \text{(c)}\ \ f(x) = \frac{x^2-4}{x+1}.$$

Question 2: Which of the following functions are even functions?

$$\text{(i) } x^4+3x^2-1; \qquad \text{(ii) } 4x^3+2x-6; \qquad \text{(iii) } |x|+x^2;$$
$$\text{(iv) } e^{-x}; \qquad \text{(v) } x^4+(x+2)^2-4$$

Question 3: Which of the following statements is true?

(i) The graph of $f(x+2)$ is obtained by shifting the graph of $f(x)$ to the left by two units.
(ii) The graph of an odd function is symmetric with respect to the y-axis.
(iii) The graph of every quadratic function crosses the x axis twice.
(iv) The graph of every function crosses the y-axis exactly once.
(v) An increasing function can cross the x-axis at most once.

Question 4: Find the domain of definition of each of the following functions.

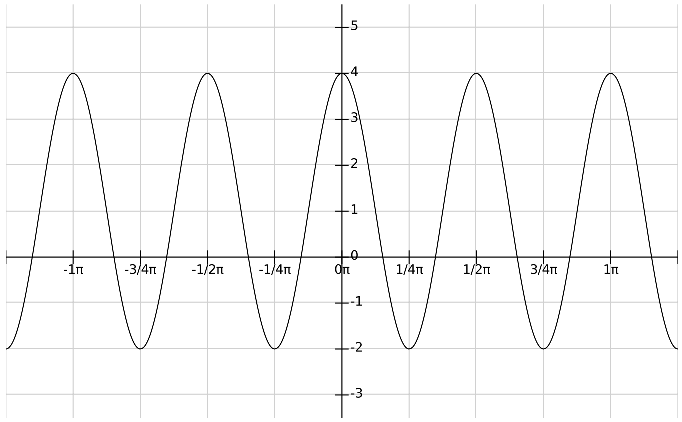
$$\text{(a)}\ \ f(x) = \frac{x+1}{x^2-2} \qquad \text{(b)}\ \ f(x) = \frac{x+2}{x(x^2-3)} \qquad \text{(c)}\ \ f(x) = \frac{x+3}{x^2-5}$$

Question 5: Find the value of the composition.

1. If $f(x) = 5x-1$ and $g(x) = -3x+2$ what is $(f \circ g)(1)$?
2. If $f(x) = 5x-1$ and $g(x) = 3x-4$ what is $(f \circ g)(1)$?
3. If $f(x) = 7x-4$ and $g(x) = -3x+1$ what is $(g \circ f)(1)$?

Question 6: Suppose we have a function $y = f(x)$ and we want to shift its graph up by 4 units and to the left by six units. How do we have to choose a and b in $y = f(x+a)+b$ to accomplish that?

Question 7: The following is the graph of a function $y = f(x)$ of the form $f(x) = M + A\cos(\frac{2\pi}{T}(x - \varphi))$.



What are the values of its mean M, amplitude A, period T and phase φ?

$\sim$

Now go and try your re-discovered pre-calculus skills online. There are various test sites where you can get even more practice problems for free. (If some questions focus on topics very different from those covered here, you may skip them as they will not be central to this course.)

The uOttawa diagnostic tool
http://science.uottawa.ca/mathstat/en/why-study/student-resources/diagnostic-test

Some US schools:

- https://math.berkeley.edu/courses/archives/placement-exam
- https://mathtesting.ucsd.edu/testing/readiness/calc.html

Evaluate yourself: Calculus readiness

1. How do I rate my Calculus background preparation?

   ☐ Excellent ☐ OK ☐ Poor

2. How do I rate my readiness to solve problems where a step may include:

   (a) manipulating algebraic expressions great OK poor
   (b) solving an equation great OK poor
   (c) solving an inequality great OK poor
   (d) using logarithms great OK poor
   (e) simplifying exponentials and logarithms great OK poor
   (f) sketching a polynomial function great OK poor
   (g) sketching an exponential or logarithmic function great OK poor
   (h) transforming the graph of a function great OK poor
   (i) finding the domain of a function great OK poor
   (j) using the absolute value function great OK poor

3. What evidence did I use to assess my readiness?

   ☐ MapleTA questions
   ☐ The practice problems
   ☐ Questions from the textbook
   ☐ Online placement tests
   ☐ Comparing myself to my friends
   ☐ My high school grades
   ☐ By how the material made sense in class
   ☐ By solving problems with friends

4. If there is any topic where I don't feel ready, do I believe I could learn it and improve?

   ☐ Yes ☐ Maybe ☐ No ☐ Not applicable

5. If there is any topic where I don't feel ready, what am I going to do about it?

   ☐ nothing
   ☐ something, when I have time
   ☐ I'll reread my notes
   ☐ I'll retry the same problems till I get the right answer
   ☐ I'll try new problems until I understand
   ☐ I'll ask my questions in the DGD
   ☐ I'll go to the math help centre with problems I couldn't solve
   ☐ I'll ask the prof questions in office hours

(Answer key, as well as more information about Self-Regulated Learning and Growth Mindsets, on page 67.)

3 Discrete-time dynamical systems (DTDS) - part I

GOAL: Find a mathematical description for dynamic processes in nature.

IDEA: Project forward from one generation to the next. A function describes how the initial measurement and the final measurement in an experiment are related. Then iterate.

Definition: A *DTDS* is an assignment of the form $x_{t+1} = f(x_t)$, where x_t is the state of the system at time t and f is the updating function. An *initial condition* is a value x_0. Sometimes a DTDS is called a recursion.

Examples:

- Constant addition between generations (e.g. trees): $x_{t+1} = x_t + c$
- Constant multiple between generations (e.g. bacteria): $x_{t+1} = rx_t$
- Constant decay and addition (e.g. medication): $x_{t+1} = rx_t + c$ with $0 < r < 1$

Definition: A *solution* of a DTDS is a sequence of numbers that describes the future of the system, i.e. $\{x_0, x_1, x_2, \dots\}$ with the property that $x_{t+1} = f(x_t)$.

Note: A solution is not a single number but an entire infinite sequence. A solution depends on the initial condition.

General Result: The general solution of a linear DTDS

$$x_{t+1} = rx_t + c,$$

with initial condition x_0 can be written in several different but equivalent forms as

$$\begin{aligned} x_t &= r^t x_0 + c(1 + r + r^2 + r^3 + \dots r^{t-1}) \\ &= r^t x_0 + \frac{r^t - 1}{r - 1} c \\ &= r^t \left(x_0 - \frac{c}{1-r} \right) + \frac{c}{1-r} \end{aligned}$$

Note: The second and third form are only valid when $r \neq 1$. We can write things even simpler by setting $x^* = \frac{c}{1-r}$. Then the general solution is

$$x_t = r^t (x_0 - x^*) + x^*$$

Observations:

1. Suppose that you have an updating function, but you can only observe your system every other generation. Then $x_{t+2} = f(x_{t+1}) = f(f(x_t)) = (f \circ f)(x_t)$. Hence the composition of updating functions corresponds to an observation two generations ahead.

2. Suppose that you missed an observation and want to infer the state in the previous generation from the next. Then $x_t = f^{-1}(x_{t+1})$ if the inverse function exists. Hence, the inverse updating function corresponds to going backwards in time.

3.1 Practice makes progress

This material is in section 3.1 of the book (2.1 in the first edition). Suggested exercises are

- Second edition 3.1:
  Drill questions: 1-14, 17–31; Applications: 32, 33, 36, 37, 41-45 ; Advanced: 58-61
- First edition 2.1: 1–12, 15–29, 61

Question 1: Suppose someone has three drinks of alcohol that bring the alcohol content in their body to 42 grams. Then the person stops drinking. Each hour, 45% of the alcohol are eliminated from the body.

1. Write the DTDS for the amount of alcohol in the body on an hourly basis.
2. Identify the initial condition and give the general solution.
3. If the amount of alcohol in the body has to be below 8 grams before one can drive, how long does the person have to wait before they can drive?

Question 2: Assume that the dynamics of caffeine absorptions are given by $C_{t+1} = 0.87C_t$, where t is time in hours and C_t is the concentration of caffeine. If $C_0 = 1000$, estimate the time needed for 80% of the caffeine to be eliminated from the body (i.e. 20% left).

Question 3: A population of rabbits is growing at a rate of 10% per year. Write down the discrete dynamical system that describes the evolution of the rabbit population. If the population is initially 1000, how many years will it take for the population to exceed 100000?

Question 4: A city is growing at a rate of 1% per year and initially has 1,000,000 individuals. What will be the population in 50 years' time?

Question 5: Suppose that a patient receives a daily dose of 50mg/L of a certain drug such that 42% of it is eliminated from the body each day. If on a certain Monday, the concentration of the drug measured in their body (shortly after the daily dose) is 55 mg/L, write down the Discrete-Time Dynamical System describing the dynamics of the concentration x_t of the drug in the body (in mg/L, t days after that Monday). Then write down the general solution to this DTDS.

4 DTDS - part II

GOAL: Visualize the behavior of a DTDS, identify special points

Graphing I: Once we have calculated a solution, we can graph it as a sequence of points $(0, x_0)$, $(1, x_1)$, $(2, x_2)$,... (t, x_t),....

Graphing II: *Without* calculating a solution, we can visualize a solution of a DTDS by using a technique called **cobwebbing**. The recipe for cobwebbing is as follows.

1. Graph the updating function.
2. Starting from $(x_0, 0)$, draw a line *vertically* to the graph of the updating function.
3. Starting from the point $(x_0, f(x_0))$ on the graph of the updating function, draw a line *horizontally* to the diagonal.
4. Repeat the previous two instructions
5. In a second plot with aligned horizontal axes, record the points (t, x_t).

Definition: A constant solution of a DTDS is called a *fixed point* or *steady state* or *equilibrium*. A fixed point is where the updating function intersects the diagonal. To calculate a fixed point algebraically, one solves the equation $x = f(x)$ for x.

Example: For the linear DTDS $x_{t+1} = rx_t + c$ the fixed point is $x^* = \frac{c}{1-r}$, when $r \neq 1$.

4.1 Practice makes progress

This material is in section 3.2 of the book (2.2 in the first edition). Suggested exercises are

- second edition: 3.1: 17-20; 3.2: 1–12, 17–35, 43
- first edition: 2.1: 15-18; 2.2: 1–12, 13-16, 17-34, 43

Question 1: A population of butterflies lives on a meadow, surrounded by forest. We want to investigate the dynamics of the population. We denote the number of butterflies at the beginning of season t by x_t. Over the course of a season, 30% of the butterflies that were there at the beginning die. During each season, 20 new butterflies arrive from other meadows.

1. Write the DTDS for the number of butterflies. What is the updating function?
2. Starting with 40 butterflies in season 0, calculate their number in seasons 1, 2, 3.
3. Calculate the fixed point of the DTDS.
4. Write the solution of the DTDS in terms of a general initial condition x_0.
5. Draw the cobweb for this DTDS, starting at $x_0 = 40$. Also draw the solution as a function of time.
6. Suppose that through some conservation measures, we can improve the quality of the pond and reduce the death rate of the butterflies. To which level do we have to reduce the death rate if we want the steady state butterfly population to be 100?

Question 2: Consider the nonlinear DTDS $x_{t+1} = \frac{rx_t}{1+0.2x_t}$, where $r > 0$ is some parameter.

1. Calculate the fixed point(s). (Note: the answer may contain parameter r.)
2. Now set $r = 3$. Calculate x_1, x_2, x_3 starting form $x_0 = 8$.
3. Keeping $r = 3$, calculate x_1, x_2, x_3 starting form $x_0 = 12$.
4. Do you observe a trend in these values? If so, what is it?

Question 3: In a forest in Alberta, every year 20% of the population of red deer either die of natural causes or are eaten by predators. In the meantime, there are 1000 new red deer. The discrete-time dynamical system that gives the population of red deer each year is $p_{t+1} = 0.8p_t + 1000$.

1. If there are 2000 red deer now, how many red deers will be there three years later?
2. Give the updating function of the dynamical system. Find its inverse, if it exists.
3. Determine all equilibrium points of the dynamical system.
4. Find the general solution of the dynamical system (ie., a formula in terms of t) given the initial condition $p_0 = 2000$.

5. Draw the solution of the dynamical system with $p_0 = 2000$ (four points are enough).
6. Draw the cobweb diagram of the dynamical system with $p_0 = 2000$ (four iterations are enough).

Question 4: Consider the discrete-time dynamical system $N_{t+1} = \frac{rN_t}{N_t-3}$, where r is a parameter. Find all values (if they exist) of parameter r, for which the system has (i) no equilibria, (ii) one equilibrium, (iii) nonnegative equilibria.

Question 5: For which value(s) of r does the DTDS $N_{t+1} = \frac{rN_t}{1+N_t}$ have a positive equilibrium?

Question 6: Suppose you deposit \$1000 each week into a special savings account, but the bank takes 5% of the total in fees. A discrete-time system modelling your investment is $x_{n+1} = 0.95x_n + 1000$.

1. If you initially have \$1500 in the bank, how much money will you have after the third week?
2. Write down the updating function of the dynamical system.
3. Find all equilbrium points of the dynamical system.
4. Give the solution of the dynamical system with $x_0 = 1500$.
5. Draw the solution of the dynamical system with $x_0 = 1500$. (Four points are enough.)
6. Draw the cobweb diagram of the dynamical system with $x_0 = 1500$. (Four iterations are enough.)

Question 7: A disease is spreading through campus. Each day, the number of people infected depends on how many were infected the day before, according to the formula $y_{n+1} = \frac{6.5y_n}{1+0.01y_n}$.

1. If one person is initially infected, how many people are infected after the first day? after the second day? after the third?
2. Find all equilibrium points of the dynamical system.
3. How many people would you guess will be infected eventually?

Question 8: Suppose that every morning a patient receives the same dose of drug. From the dose, the drug concentration in his blood increases by 2. Over the course of 24 hours between doses, 75% of the drug in the blood is removed.

1. Write the linear DTDS for the drug concentration, $x_{t+1} = f(x_t)$, and find x_4 when $x_0 = 88$.
2. Draw the updating function and start the cobwebbing process at $x = 0.2$.
3. Find the fixed point explicitly.

Question 9: In order to keep the songbirds in the back yard happy, one person puts out 20g of seeds at the end of each week. During the week, the birds find and eat 2/3 of the available seeds. The DTDS for the amount of seeds in the back yard is $S_{t+1} = 1/3S_t + 20$, where t is measured in weeks and seeds are counted just before a new supply is provided.

1. What is the updating function of the DTDS?
2. Find the fixed point of the DTDS if there is one.
3. Find the general solution formula for the DTDS, i.e., $S_t = ...$
4. Graph the updating function and draw the cobwebbing, starting from $S_0 = 5$, for at least 4 steps.

Question 10: A group of patients is given a certain dose of a drug once. Two measurements of concentration of the drug in the blood are taken 24 hours apart to determine the rate at which the drug is removed from the blood stream. The measurements are given below.

patient	initial measurement	final measurement
1	3	1
2	4.5	1.5
3	0.6	0.2
4	1.8	0.6

1. Write a DTDS of the form $x_{t+1} = ax_t$ for drug removal and find the value of a.
2. For patient 1, how long will it take until the drug concentration is below 0.1?
3. How long does it take for the initial concentration to decrease by 50%?
4. Now patients are given a dose every 24 hours, i.e., we have the DTDS $x_{t+1} = ax_t + b$ with a as in part (a). How much of the drug has to be given so that the steady state concentration is 6?

Question 11: 7. (7 points) The population of fish in Fisher's Pond grow at a steady rate annually, but fishing is so popular that, without restocking, the population would die out. Therefore Fisher's Pond is restocked with fish each spring. Using historical data, we declare that DTDS modeling the population of fish, with x_t representing the average number of fish per m^2 of surface area in year t, is given by

$$x_{t+1} = 0.7x_t + 4.8.$$

(a) Find the fixed point x^* of this DTDS.

(b) Suppose that in year zero there were 36 fish/m^2. Give the general solution formula to this DTDS.

(c) Determine the (whole) number of years necessary until the number of fish per m^2 is within 0.5 of the fixed point, that is, until $|x_t - x^*| < 0.5$.

5 DTDS - part III

GOAL: Analyze DTDS, and learn more complicated examples.

Examples: Cobweb a linear DTDS with (i) $x_{t+1} = 1.5x_t$, (ii) $x_{t+1} = 0.5x_t$, (iii) $x_{t+1} = -0.5x_t$, (iv) $x_{t+1} = -1.5x_t$. Notice that in cases (ii) and (iii) all solutions approach the only steady state $x^* = 0$, whereas in cases (i) and (iv), solutions grow in absolute value.

Definition: A fixed point is called *stable* if all nearby solutions approach the point, and *unstable* if at least one nearby solution does not approach the point.

Obervation: In the linear DTDS $x_{t+1} = rx_t + c$, the solution formula

$$x_t = r^t(x_0 - x^*) + x^*$$

makes it clear that x_t approaches x_0 exactly when $|r| < 1$. So we say that the fixed point x^* is stable exactly when $|r| < 1$. There are four qualitatively different cases for cobwebbing according to (i) $0 < r < 1$, (ii) $r > 1$, (iii) $-1 < r < 0$, and (iv) $r < -1$.

Note: A stable fixed point gives information about the long-term behavior of a DTDS. An unstable fixed point would not be observed in nature, but still carries important information.

Nonlinear updating functions Most processes in nature are not linear. Therefore, we need nonlinear updating functions to describe such processes. There is in general no explicit way to write down a general solution. But cobwebbing works. In each of the following examples, find steady states analytically and use cobwebbing to determine their stability.

1. Beverton-Holt updating function $f(x) = \frac{rx}{1+\alpha x}$
2. Ricker updating function $f(x) = rx\exp(-\alpha x)$
3. Allee updating function $f(x) = \frac{ax^2}{b^2+x^2}$
4. Alcohol adsorption (see book)
5. Heartbeat model (see book)

5.1 Practice makes progress

This material is in sections 3.2-3.4 of the book (2.2, 2.3 and 2.5 in the first edition). Suggested exercises are

- second edition: **3.2:** 13-16, 23-30, 35-40; **3.3:** 1-19, 27-29, 30-34; **3.4:** 11-18, 39-42
- first edition: **2.2:** 13-16 **2.3:** 1-19, 27-29, 28-32; **2.5:** 11-18, 39-42

Question 1: Apply the stability criterion to all the practice problems with linear updating function in the previous section and find whether the steady state is stable.

Question 2: Consider the discrete-time dynamical system (DTDS) $M_{t+1} = -0.8M_t + 6$.

(a) Find the updating function of the DTDS.

(b) (b) Find the equilibrium point of the DTDS.

(c) Give the general solution formula for the DTDS.

(d) Calculate M_{10} if $M_0 = 0$.

(e) Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.

(f) Is the equilibrium point stable or unstable?

Question 3:

(a) Because of high mortality and low reproductive success some fish species experience exponential decline over many years. Atlantic Salmon in Lake Ontario, for example, declined by 80% in the 20-year period leading up to 1896. Denote the number of Atlantic Salmon in Lake Ontario in year t by x_t and write the equation $x_{t+1} = rx_t$. Calculate the value of r.

(b) Due to fishing restrictions, the value of r changed and the population is less at risk now. The major reason for the recovery of Atlantic Salmon, however, is a massive restocking program in Ontario. The population dynamics can now be described by the DTDS $x_{t+1} = 0.2x_t + c$, where c is the number of fish restocked every year.

  (i) What is the updating function of this DTDS ?

  (ii) What is the equilibrium point of this DTDS ?

  (iii) Is the equilibrium point stable or unstable for the DTDS? Why?

(c) Now we assume that there are 1000 fish restocked annually.

  (i) Find the general solution formula for the DTDS with this value.

(ii) Draw the cobweb, starting from $M_0 = 1000$ for at least 4 steps.

(iii) How does the number of restocked fish need to be adjusted to ensure an equilibrium population of 1500 fish?

Question 4: A patient receives a daily dose $d = 5$mg of the drug FilGud®. In the course of 24 hours, 60% of the drug is absorbed and a fraction of $p = 0.4$ remains in the blood. The DTDS modelling the daily concentration M_t of FilGud® in the blood immediately after administering the dose is $M_{t+1} = pM_t + d = 0.4M_t + 5$.

(a) Find the updating function of the DTDS.

(b) Find the equilibrium point of the DTDS.

(c) Give the general solution for the DTDS with general initial condition M_0.

(d) Calculate M_5 if $M_0 = 0$.

(e) Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.

(f) Is the equilibrium point stable or unstable?

(g) Due to sudden complications, the patient now also needs to take the drug WelSun®. This drug inhibits the uptake of FilGud® so that only 50% will be absorbed and a fraction of $\tilde{p} = 0.5$ will remain in the blood. Calculate the new daily dose $\tilde{d}$ of FilGud® needed to maintain the equilibrium concentration of that drug at the same level as before.

Evaluate yourself: DTDS

1. What is a DTDS? Give both the mathematical definition, and also an explanation.

2. What is the difference between the updating function and the DTDS?

3. What is a cobweb diagram? What does it do for us?

4. What is the difference between the cobweb and the solution?

5. Does every DTDS have a solution?

6. Does every DTDS have that nice general solution formula?

7. What is the difference between a linear and a nonlinear DTDS?

8. How do you choose the updating function for a DTDS?

9. What is the difference between a fixed point, a steady state, and an equilibrium?

10. Describe some ways you can find a fixed point of a DTDS.

11. How do you decide if a fixed point is stable or not?

12. Why do we say that you don't observe unsteady fixed points in nature?

(Answer key on page 67.)

6 Limits of functions

GOAL: Characterize the behavior of a function near a point where the function might not be defined.

Definition: Limit of a function. We say that the limit of a function f as x approaches a equals L and write

$$\lim_{x\to a} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x very close to a.

Observations:

- The value of $f(a)$, if it even exists, does not matter in the definition of a limit.
- We can define one-sided limits in a similar way and denote them by $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$.

Definition: Existence of a limit. We say that the limit of f as x approaches a exists if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Observations:

- Calculators might fail at finding limits. In this class, we use other means.
- Identify limits when a graph is given.
- Algebraic rules to find limits.

A combination of the following rules together with algebraic manipulations of the expressions allow us to calculate many limits.

Limit laws: The following are true.

1. $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$.

If the limits involved exist, then the following hold.

2. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ provided $\lim_{x\to a} g(x) \neq 0$

Direct Substitution Rule: If the function is a polynomial, a rational function, a trig function, an exponential or logarithm function, or a root function, or a composition of these, and if a is in the domain of the function, then $\lim_{x\to a} f(x) = f(a)$, in particular the limit exists.

6.1 Practice makes progress

The material is covered in Section 4.2 of the book (3.2 in the first edition) with Section 4.1 (3.1) being one motivation to study such things as limits. Suggested exercises are

- second edition: **4.1**: 1–26, **4.2:** 1–7, 10–13, 30–55
- first edition: **3.1**: 1–26, **3.2:** 1–7, 10–13, 30–55

Question 1: Does the limit $\lim_{x\to 4} \dfrac{x-\sqrt{3x+4}}{4-x}$ exist? If so, what is its value?

Question 2: Let $G(x) = \dfrac{x^2-5x+6}{|3-x|} + |x-2|$.

a) Find $\lim_{x\to 3^+} G(x)$.

b) Find $\lim_{x\to 3^-} G(x)$.

c) Does $\lim_{x\to 3} G(x)$ exist?

Question 3: For a real number b, consider the function

$$f(x) = \begin{cases} \sin(x-b), & x > 0 \\ x^2+1, & x < 0. \end{cases}$$

Find the smallest possible positive value of b such that the limit $\lim_{x\to 0} f(x)$ exists.

Question 4: Use your calculator to guess whether the limit $\lim_{x\to 0}\left(\dfrac{e^x-1}{x}\right)$ exists and what its value might be. (Yes, this is one of the few problems where you are allowed to use a calculator. The point is that you remember to choose values on both sides of the point $x = 0$ in your sequences.)

Question 5: Calculate the following limits, if they exist, or explain why they do not exist. Justify your answers without using sequences of numerical values for x.

$$\lim_{x\to 1} \frac{1-x^2}{x^2-3x+2} \qquad \lim_{x\to 0} \frac{3x^3}{\sqrt{5x^6-4x}}$$

$$\lim_{x\to 0} \frac{\sin^2(x)}{\cos^2(x)} \qquad \lim_{t\to 0} \left(\frac{1}{t\sqrt{1+5t}} - \frac{1}{t}\right)$$

$$\lim_{t\to 0} \left(\frac{t}{1-\sqrt{1+t}}\right) \qquad \lim_{x\to 4} \frac{3-\sqrt{25-x^2}}{x-4}$$

Question 6: Let g be defined by $g(x) = \dfrac{x^2-9}{|x-3|}$.

a) Calculate $\lim_{x\to 3^+} g(x)$.

b) Calculate $\lim_{x\to 3^-} g(x)$.

c) Does the limit $\lim_{x\to 3} g(x)$ exist?

Question 7: Let g be defined by $g(x) = \dfrac{x^2-1}{|x-1|}$.
a) Calculate $\lim_{x\to 1^+} g(x)$.
b) Calculate $\lim_{x\to 1^-} g(x)$.
c) Does the limit $\lim_{x\to 1} g(x)$ exist?

Question 8: For which number a does the limit $\lim_{x\to 0} f(x)$ exist, where

$$f(x) = \begin{cases} e^{x-a}-1, & x > 0 \\ x^2+1, & x < 0. \end{cases}$$

Question 9: (a) Find the following limit without using a calculator

$$\lim_{x\to -2} \frac{|x+3|-1}{x^2-4}$$

(b) Does the following limit exist? If yes, give the limit, if not, justify your answer. If you need a calculator to work out the answer, give at least 4 values of x that you tried.

$$\lim_{x\to 2} \frac{|x+3|-1}{x^2-4}$$

7 Limits, infinity, and continuity

GOAL: Extend the material from the previous class to infinity; introduce continuity.

Definition: Infinite limits. We say that the limit of a function f as x approaches a is infinity (or negative infinity) if we can make $f(x)$ as large (negative and large) as we wish by choosing x very close to a. We write

$$\lim_{x\to a} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to a} f(x) = -\infty.$$

This definition applies also to one-sided limits. Geometrically, an infinite limit at a finite value a corresponds to a *vertical asymptote* of the graph at $x = a$.

NOTE: Infinity is *not* a number.

We have relatively few tools to prove infinite limits. Examples are rational functions; and remember that if $g(x) \to 0$ as $x \to a$ then $1/g(x) \to \pm\infty$ as $x \to a$.

Definition: Limits at infinity We say that the limit of a function f as x approaches ∞ equals L and write

$$\lim_{x\to\infty} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x arbitrarily large. We can also consider the limit $x \to -\infty$, but obviously, these limit can only be one-sided. A limit at infinity corresponds to a *horizontal asymptote* of the graph at $y = L$.

These limits at infinity can be evaluated with the same algebraic procedures as limits at finite values a, using "arithmetic with infinity":

$$\infty + c = \infty; \quad c \times \infty = \infty; \quad \frac{c}{\infty} = 0; \quad \frac{1}{0+} = \infty; \quad \frac{\infty}{c} = \infty$$

where $c > 0$ is a constant nonzero number. However, expressions such as $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are called *indeterminate forms* and getting one means you have to keep simplifying and using other methods to find the limit.

Definition: Continuity A function f is called *continuous at a point a* if $f(a)$ exists, and if the limit equals that value, $\lim_{x\to a} f(x) = f(a)$. In particular, this limit needs to exist. A function is called *continuous on an interval* if it is continuous at every point in that interval.

Examples: polynomials, rational functions (where defined), trigonometric and inverse trigonometric functions (where defined), exponentials and logarithms are continuous. The sign function is not continuous at $x = 0$.

Note: Look out for when checking continuity: division by zero, log of zero, piecewise defined functions at places where the function definition changes.

Theorem: Continuity and exchanging limits. If f is continuous and g is a function so that $\lim_{x\to a} g(x) = b$ exists, then we can take the limit of the composition

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

7.1 Practice makes progress

The material is covered in Sections 4.3/4.4 of the book (3.3/3.4 in the first edition). Please note that we are not covering "Comparing functions at infinity" (this topic will come back to us later) not will we talk about "Limits of sequences". In the section on continuity, we place little emphasis on questions of "input/output precision". Suggested exercises are

- second edition: **4.3:** 1–14, 18–26, 27–30, 31–50
- second edition: **4.4:** 1–29, 24–43
- first edition: **3.3:** 1–14, 18–24, 25–48, 50, 51
- first edition: **3.2:** 1–29, 24–43

Question 1: Calculate the following limits:

$$\lim_{x\to\infty} \frac{x^2 - \frac{3}{2}x - 1}{x-2} \qquad \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+2}} \qquad \lim_{x\to\infty} \frac{3x^3}{\sqrt{5x^6-4x}}$$

$$\lim_{x\to 1^+} \frac{1+x^2}{x^2-3x+2} \qquad \lim_{x\to 0} \frac{1}{x}\left(\frac{1-x}{1+x}\right) \qquad \lim_{t\to 0} \left(\frac{1}{t\sqrt{1+5t}} - \frac{1}{t}\right)$$

Question 2: Does the limit $\lim\limits_{x\to-\infty} e^{1/x}$ exist? If so, what is its value?

Question 3: Consider the function $f(x) = \dfrac{1+2e^{-x}}{1-e^{-x}}$.
(a) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
(b) Are there any vertical asymptotes? If yes, find the left and right hand limit in each case.

Question 4: Let $f(x) = \dfrac{x^2-4}{|x-2|}$. Calculate $\lim\limits_{x\to 2^+} f(x)$ and $\lim\limits_{x\to 2^-} f(x)$ Is f continuous at $x = 2$?

Question 5: Is the following function continuous at $x = 1$? Justify your answer in a short sentence.

$$f(x) = \cos(2x) + \frac{3x^2-5x}{x^2-2}$$

Question 6: Can one choose a value for a such that the following function is continuous at $x = 3$? If yes, what is the value and why? If no, why not?

$$f(x) = \begin{cases} \frac{|x-3|}{x^2-9}, & x \neq 3 \\ x+a, & x = 3. \end{cases}$$

Question 7: Let $f(x) = \begin{cases} \dfrac{x^2-4x+3}{(x-1)^3} & \text{if } x \neq \pm 1, \\ x+b & \text{if } x = \pm 1. \end{cases}$, with b a real number

(a) Find $\lim\limits_{x\to 1} f(x)$.

(b) Is there a value of b that makes the function f continuous at $x = 1$?
(c) Find $\lim_{x\to -1} f(x)$.
(d) Is there a value of b that makes the function f continuous at $x = -1$? If yes, then provide the value.

Question 8: For what values of a and b is the following function continuous everywhere?

$$f(x) = \begin{cases} a\sin(x) + b, \ x \leq \ 0 \\ x^2 + a, \ 0 < x \leq \ 1 \\ b\cos(2\pi x) + a, \ x > 1 \end{cases}$$

Question 9: Find the value of a so that the following function is continuous

$$f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & x \neq 1 \\ a & x = 1. \end{cases}$$

Question 10: Consider the function $g(x) = \begin{cases} \frac{a}{(\sin(x))^2+1} & \text{if } x < \frac{\pi}{2} \\ \frac{kx+1}{x+1} & \text{if } x \geq \frac{\pi}{2} \end{cases}$

(a) What is the condition on a and k such that g is continuous at $\pi/2$?
(b) Find a and k so that g is continuous and has the horizontal asymptote $y = 2$ as $x \to \infty$.

Question 11: For which value of the parameter a is the following function continuous at $x = 3$?

$$f(x) = \begin{cases} a\cos(\pi x) & \text{if } x < 3 \\ \frac{x}{2} - 4 & \text{if } x \geq 3. \end{cases}$$

Evaluate yourself: Midterm #1 check-in

1. What are my goals? (Long-term, short-term, whatever.)

2. How does MAT1330 fit in towards achieving those goals?

3. What are my specific goals for this course? I think of both the outcome (a grade, knowledge, skills) and of the process to get there (organization, discipline, consistency, stick-to-it-iveness).

4. How am I doing towards achieving my specific goals for MAT1330?

☐ Very well ☐ OK ☐ Poorly

5. What can I do (action plan!) to ensure I achieve my goals?

(Answer key on page 68.)

8 Differentiability

GOAL: Define the slope of a function at a point.

Idea: We know what the slope of a line is. Given two points, $x \neq y$ in the domain of function f, we write the slope of the secant line though the two points as $\frac{f(y)-f(x)}{y-x}$. Now we take the limit as $y \to x$ and define this limit to be the slope of the tangent line.

Definition: A function f is called *differentiable at a point* x if the limit

$$f'(x) := \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$$

exists. (Note that f has to be defined and continuous at x.) A function is called differentiable on an open interval, if it is differentiable at each point of the interval.

Note: Failure to be differentiable can correspond to a corner, a cusp, a vertical asymptote, a doscontinuity.

Derivatives and Graphs: We have a correspondence

$$\begin{gathered} f'(x) > 0 \Leftrightarrow f \text{ increasing at } x \\ f'(x) = 0 \Leftrightarrow f \text{ has horizontal tangent at } x \\ f'(x) < 0 \Leftrightarrow f \text{ decreasing at } x \end{gathered}$$

Definition: x is called a *critical point of* f if x is in the domain of f and either $f'(x) = 0$ or $f'(x)$ is not defined.

Note: Calculation of derivative from the definition/from first principles can be done in simple cases. But in general, we want to have rules that we can use faster, rather than going back to the definition every time.

Differentiation rules: If the derivatives exist, then the following rules hold

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
2. $(f \pm g)'(x) = f'(x) \pm g'(x)$
3. If $h(x) = f(x) \cdot g(x)$ then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. If $h(x) = \frac{u(x)}{v(x)}$ then $h'(x) = \frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}$

8.1 Practice makes progress

The material is covered in Sections 4.5/5.1/5.2 of the book (3.5/4.1/4.2 in the first edition). The second edition covers the derivative of the exponential function already here. The first edition has it later. We will also treat it later. Suggested exercises are

- second edition: **4.5:** 8, 9, 15–17, 22–28, 39–43, 44–49, 50–53, 54–57
- second edition: **5.1:** 1–42, **5.2:** 1–43 (the problems with the exponential function are for the next lecture)
- first edition: **3.5:** 8, 9, 15–17, 22–28, 39–43, 44–49, 50–53, 54–57
- first edition: **4.1:** 1–16, **4.2:** 1–19

Question 1: Use the definition of the derivative to calculate the derivative of the functions.

$$\text{(a) } f(x) = 1 + \sqrt{2x+3} \qquad \text{(b) } g(x) = \frac{x}{1+x} \qquad \text{(c) } h(x) = \sqrt{x^2-1}$$

$$\text{(d) } f(x) = \frac{2}{2015-x} \qquad \text{(e) } g(y) = \frac{2}{3+4y} \qquad \text{(f) } h(z) = (z-1)^2 + 3z - 12$$

Question 2: Use the rules from class to calculate the derivatives of the following functions.

$$\text{(a) } g(x) = \frac{1+x^2}{\sqrt{x}+x^{-1}} \qquad \text{(b) } f(x) = 12x^{12} - 11x^{11} + 10x^{10}$$

$$\text{(c) } h(z) = \frac{1}{z^3} + \frac{1}{z^{1/3}} \qquad \text{(d) } f(x) = 15y^2 - 3y + 1$$

Question 3: Give an example of a function that is continuous everywhere but not differentiable at $x = 2$.

Question 4 (hard!): Consider the function

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ x^2, & x \neq \mathbb{Q}. \end{cases}$$

Is this function continuous at $x = 0$? Is it differentiable at $x = 0$?

9 Differentiating exponentials, logarithms, and the chain rule

GOAL: Learn how to differentiate these important functions, and compositions of functions in general.

The exponential function:

$$\frac{d}{dx}e^x = e^x.$$

Note: The exponential function $f(x) = ke^x$ for any real number k is the *only* function with the special property $f'(x) = f(x)$.

Examples: Find the derivatives with the rules that we have learned so far.

- $f(x) = e^{3x}$ (use repeated product rule)
- $f(x) = e^{-x}$ (use quotient rule)
- $f(x) = x^n e^{-x}$. This function is related to the Gamma function; it is quite important in statistics.

The Chain Rule: If f, g are differentiable functions, then

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x))g'(x).$$

Special case: Inverse functions.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Application: Logarithm: The inverse of $f(y) = e^y$ is $f^{-1}(x) = \ln(x)$. Then we calculate

$$(\ln(x))' = \frac{1}{(e^y)'} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

Using logarithms in differentiation:
Now that we know the derivative of the natural logarithm, we can use it to differentiate other functions. The important formula is $a = e^{\ln a}$.

- To differentiate $f(x) = \log_a x$, we write $f(x) = \frac{\ln x}{\ln a}$ and differentiate simply $f'(x) = \frac{1}{x \ln a}$.
- To differentiate $f(x) = x^x$, we rewrite $f(x) = e^{x \ln x}$ and use the chain and the product rule.

$$f'(x) = e^{x \ln x}\frac{d}{dx}(x \ln x) = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1).$$

9.1 Practice makes progress

The material is covered in Sections 5.1 and 5.3 of the book (4.3 and 4.4 in the first edition). Suggested exercises are

- second edition: **5.1:** 1–42 (*All* the problems with the exponential function)
- second edition: **5.3:** 1–14, 17–41, 42–57
- second edition: sketching problems **5.3:** 5, 16
- second edition: applications **5.3:** 62–68
- first edition: **4.3:** 1-27, 36-41
- first edition: **4.4:** 1–46

Question 1: For each of the following functions, find the derivative.

(a) $f(x) = \ln\left(e^{x^3} \cdot \dfrac{x^4 - 3x^2 + 17x - 8}{\ln(x)}\right)$

(b) $f(x) = \ln\big(1 + x\ln(x)\big)$

(c) $f(x) = \left(\dfrac{\sqrt{x+1}}{e^{x^2} + 5}\right)^8$

(d) $y(x) = \log(10^x(3x^2 - 1)^3)^5$

(e) $f(x) = \ln(x^{10})$

(f) $h(z) = \dfrac{5^{-z}}{\sqrt{z}}$

(g) $f(x) = \dfrac{1}{\sqrt{2}} e^{-(x-10)^2}$

(h) $f(x) = \ln\left(\dfrac{1}{x^2 + 1}\right)$

(i) $y(x) = \ln\left(7e^{x^3}(x^3 - 3)^2\right)$

(j) $w(y) = \ln\left(\sqrt{y^2 + 3y + 9}\right)$

(k) $f(t) = \ln\left(\dfrac{(t^3 + 1)^7}{(t + 2)^6}\right)$

(l) $g(x) = \ln\left(e^x x^3\right)$. Simplify!!

(m) $h(x) = \sqrt{e^{2x} + x^3}$

Question 2: Consider the function $f(x) = e^x x^{-n}$ with parameter $n = 1, 2, 3, \ldots$. Find the equation of the tangent line to the graph of the function at $x = 1$.

Question 3: Find the equation of the tangent line to the curve

$$y = x^{1/3} - 16/x$$

at the point $(8, 0)$.

10 Sine, Cosine and implicit differentiation

GOAL: Differentiate trigonometric functions and functions that are given implicitly.

Derivatives of some trig functions:

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = 1 + \tan^2(x) = \sec^2(x)$$

Note: Mathematical notation is not always consistent. We write $\cos^n(x) = (\cos(x))^n$ for *positive* integers n. But $\cos^{-1}(x)$ stands for the inverse function $\arccos(x)$ and not for the fraction $\frac{1}{\cos(x)}$.

Inverse trigonometric functions: via the chain rule

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sin(\arccos(x))} = \frac{-1}{\sqrt{1-\cos^2(\arccos(x))}} = \frac{-1}{\sqrt{1-x^2}}.$$

Implicit differentiation: Sometimes we want to find the derivative of a function that is not given explicitly but rather in a complicated equation. We can apply the chain rule and solve for the derivative that we are interested in.

Logarithmic differentiation: When the independent variable appears in the base and in the exponent of a function, then we can take logarithms first (on both sides!) and then differentiate and solve for the derivative that we are interested in.

$$y(x) = f(x)^{g(x)} \quad \Rightarrow \quad \ln(y(x)) = g(x)\ln(f(x)) \quad \Rightarrow \quad \frac{y'}{y} = g'\ln(f) + g\frac{f'}{f}$$

10.1 Practice makes Progress

The material here is covered in Sections 5.4 and 5.5 of the book (4.5 and 4.4 in the first edition). Suggested exercises are

- second edition: **5.3:** 49–57
- second edition: **5.4:** 1–31, 32–35, 36–40, 49–56
- second edition: **5.5:** 1–19
- first edition: **4.5:** 1–31, 32–35, 45–48
- first edition: **4.4:** 35–38, 51–57

Question 1: The location (as a function of time) of a car, moving in a straight line, is given by the expression $x(t) = 2t + \sin(2\pi t)$ for $t \in [0, 1]$. What are the highest and lowest values of its acceleration in that time interval?

Question 2: Find the derivatives of the following functions. Do not simplify.

(a) $f(x) = e^{\arcsin^5(x)+\arcsin(x^5)}$

(b) $f(x) = \arctan(\cos\sqrt{x}) + \arccos(\tan\sqrt{x})$

(c) $f(x) = \left(\sqrt[3]{\sin^2(x)}\right)^{\arctan(x)}$

(d) $y(x) = \sin(e^{\sec(x^2)})$

(e) $y(x) = \dfrac{5x}{(\tan(x^2))^3}$

(f) $g(s) = \dfrac{\cos(5s+8)}{\sin(5s)}$

(g) $y(x) = \dfrac{1}{(\sin(x^2))^2}$

(h) $y(x) = \cos(e^{\cos x})$

(i) $f(x) = \cos(x)\sin(5x^2+7)$

(j) $g(x) = \dfrac{\tan(x)}{e^{7x}x^4}$

(k) $h(x) = e^{\cos^3(x)+2\sin^2(x)}$

(l) $h(z) = \dfrac{\sin(z^5)}{\sqrt{z}}$

(m) $g(x) = e^{\cos^2(x^3)}$

(n) $f(x) = \dfrac{\tan(1-x^3)}{x+1}$

(o) $g(x) = x^{\cos(x)}$

(p) $h(x) = \ln\left(x^5\sin(1+\sqrt{x})\right)$

(q) $f(x) = 2^{\sin(x)}$

Question 3: If $f(x) = e^x \arctan(x)$ then what is $f'(1)$?

Question 4: For each of the following, use implicit differentiation to find $\frac{dy}{dx}$ when x and y satisfy :

(a) $e^{4x\sqrt{y}} = y^5$.

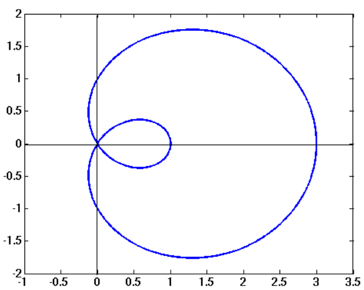
(b) $ye^y - xe^x = 0$.

(c) $y\ln(y) - x\ln(x) = 0$.

(d) $2xe^{-x^2} + 3y^2e^y = 5x$.

Question 5: A function $y = f(x)$ is defined implicitly by the equation $x^2y^7 - x^5\ln(y) = 4$.
(a) Find the derivative $\frac{dy}{dx}$
(b) Find the equation of the tangent line to the graph of this equation at the point $(2, 1)$.

Question 8: Find the tangent line to the "limax" curve (see plot below) at the point (0,1), where the curve is given by the relation $(x^2 + y^2 - 2x)^2 = x^2 + y^2$.



Question 9: The trifolium is the curve (see plot below) given by the equation $12xy^2 - 4x^3 = x^4 + 2x^2y^2 + y^4$.
Identify on the curve of the trifolium the points where we cannot compute dy/dx. Find the equation of the tangent line on the trifolium at the point $(2, 2)$.

11 Second derivative and curve sketching

GOAL: understand curvature, develop a recipe for curve sketching.

Motivation: Consider the two functions $f(x) = e^x$ and $g(x) = \frac{x}{1+x}$ for $x \geq 0$. Both functions have positive first derivatives, hence, both are increasing functions.

$$f'(x) = e^x > 0, \qquad g'(x) = \frac{1}{(1+x)^2} > 0.$$

However, their graphs look quite different. The slope of f increases with x but the slope of g decreases with x. In other words, the function $f'(x)$ is increasing but the function $g'(x)$ is decreasing. Let's calculate this:

$$\frac{d}{dx}[f'(x)] = e^x > 0, \qquad \frac{d}{dx}[g'(x)] = \frac{-2}{(1+x)^3} < 0 \quad x \geq 0.$$

Definition: The second derivative of a function is defined as

$$\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}f(x) = f''(x).$$

Second derivatives and graphs: The second derivative measures curvature. If it is positive, then the graph curves upwards (or to the left when thinking about traveling along the graph in positive x-direction). When it is negative, then the graph curves downward (or to the right). A point where the curvature changes from concave up to concave down (or vice versa) is called *point of inflection*. It necessarily has $f''(x) = 0$.

Examples: Power functions $f(x) = x^p$. There are three cases to consider: (i) $p > 1$, (ii) $0 < p < 1$, and (iii) $p < 0$.

Note: The second derivative can also help to identify critical points as (local) minima and maxima. But not always. The same caution is to be exercised with inflection points. Examples are the function $f(x) = x^3$ and $g(x) = x^4$.

Curve Sketching:

1. find the domain of the function
2. find the zeros
3. identify the asymptotes
4. differentiate and find critical points, intervals of increase and decrease
5. differentiate again and find possible inflection points, upward and downward concavity
6. sketch the x-axis and all the information
7. MAKE SURE EVERYTHING IS CONSISTENT

11.1 Practice makes Progress

The material here is covered in Sections 5.6 of the book (4.6 in the first edition). Suggested exercises are

- second edition: **5.6:** 1–8, 11–31, 36–43, 44–47, 66–73
- first edition: **4.6:** 1–8, 11–25, 26–35, 50–57

Question 1: Consider the function $f(x) = \dfrac{x-1}{x-2}$.
(a) Find the domain of f.
(b) Find the limits of f as x approaches $\pm\infty$
(c) Are there points where f is not continuous? If yes, find the left and right limits there.
(d) Find the intervals where f is increasing and decreasing. Are there critical points?
(e) Find the intervals where f is concave up or concave down.
(f) Draw the graph of f.

Question 2: Consider the function $f(x) = \dfrac{2x^2+4x+3}{x^2-1}$.
(a) Find the domain of definition of f.
(b) Find the limits of f as x approaches $\pm\infty$.
(c) Are there any infinite limits? If yes, find the left and right hand limit in each case.
(d) Find the intervals where f is increasing and where f is decreasing.
(e) Where does f have a horizontal tangent line?

Question 3: Consider the function $f(x) = \sqrt{3x}\, e^{-x/6}$.
(a) Find the domain of definition.
(b) Find the critical point(s).
(c) Find the intervals where f is increasing or decreasing.
(d) Use a table of values to guess a horizontal asymptote.
(e) Find the intervals where f is concave up or down.
(f) Sketch the graph of the function $y = f(x)$.

Question 4: Consider the function $f(x) = \dfrac{1}{x^2} + \dfrac{1}{2x^3}$. Follow these steps to graph the function.
(a) Find the domain of f.
(b) Find the x-intercept(s) of f.
(c) Calculate the derivative of f.
(d) Find the critical point(s) of f.
(e) Calculate the second derivative of f.
(f) Find the point(s) of inflection.
(g) Find the limits $\lim_{x\to 0^\pm} f(x)$.
(h) Find the limits $\lim_{x\to\pm\infty} f(x)$.
(j) Sketch the graph of f for $x \in [-2, 2]$.

Question 5: Graph the function $f(x) = \dfrac{-x^2+x+1}{1-x^2}$ using the following steps.

(a) Give the domain of $f(x)$.
(b) Find the vertical asymptotes.
(c) Find the horizontal asymptotes.
(d) Find the zeros of the function and the y-intercept.
(e) Find the first derivative and simplify it. Does $f(x)$ have any critical points?
(f) Find the inflection points. Indicate where $f(x)$ is concave up or down.
(g) Graph the function $f(x)$, including all the information found above.

Question 6: Use the first and second order derivatives to sketch the graph of $f(x) = \frac{1}{x} - \frac{2}{x^2}$. You have to find the zeros, critical points, inflexion points, intervals where the function is increasing and decreasing, and the vertical and horizontal asymptotes if any.

Question 7: Suppose a function $y = f(x), -\infty < x < \infty$, is continuous, with continuous first and second derivatives. Assume it satisfies the following conditions:

1. $f'(x) < 0$ when $x < 0$, and $f'(x) > 0$ when $x > 0$
2. $f''(x) < 0$ when $x < -2$, and $f''(x) > 0$ when $x > -2$
3. $\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = 2$.
4. $f(0) = -3, f(-2) = -1$.

(a) Where is the graph of $f(x)$ decreasing?
(b) Where is the graph of $f(x)$ concave up?
(c) Where does $f(x)$ attain a local maximum or minimum?
(d) What are the asymptotes of f?
(e) Sketch the graph of the function $y = f(x)$.

Question 8: Given that

$$f(x) = \frac{x}{x^3 - 1}, \quad f'(x) = \frac{-2x^3 - 1}{(x^3 - 1)^2} \quad \text{and} \quad f''(x) = \frac{6x^2(x^3 + 2)}{(x^3 - 1)^3},$$

find all critical points of f and of f', identify all intervals on which f is increasing, identify all intervals where the graph of f is concave up, identify all inflection points, identify all local extrema. Then sketch the graph of $y = f(x)$.

12 Extreme values

GOAL: Identify maxima and minima of functions. (Note that there are different kinds.)

Definition: An *absolute (or global) maximum* of a function f occurs at point c if $f(x) \leq f(c)$ for all x in the domain of f. An *absolute (or global) minimum* occurs if $f(x) \geq f(c)$ for all x in the domain of f. We also call global maximum and minimum *global extreme values*.

Examples: Functions may have no global max or min, they may have only one but not the other, they may have both, and such a global max or min need not be unique.

Definition: A *relative (or local) maximum* of a function f occurs at point c if $f(x) \leq f(c)$ for all x near the point c. A *relative (or local) minimum* occurs if $f(x) \geq f(c)$ for all x in an open interval around point c. We also call these *local extreme values*.

Examples: Functions may have no local max or min, they may have only one but not the other, they may have both, and a local max or min need not be unique.

Note: A point at the boundary of the domain of the function cannot be a local extremum since we cannot test all values of x near c, but only those in the domain of f.

How do we find extreme values?
Fermat's little theorem says that if f has an extreme value at c and if $f'(c)$ exists, then $f'(c) = 0$. This means that we only have to check critical points. But not every critical point is an extreme value. We need more.

1. If c is a critical point of f and if the sign of the derivative of f changes at c, then a local extremum occurs at c.
2. If c is a critical point of f and if $f''(c)$ exists and is not zero, then a local extremum occurs at c.
3. Note that if $f''(c) = 0$ we cannot conclude whether an extremum occurs at c. Note also that sometimes it is much harder to calculate f'' than to check for a sign change in f'.
4. All points where $f'(c)$ does not exist need to be checked individually.

Recipe for finding extreme values of a function: First find all critical points. Then check each critical point. Finally, compare the function value at critical points and, if applicable, endpoints of the domain of definition.

How do we know that extreme values are there?
The *extreme value theorem* says that a continuous function on a bounded interval has a global maximum and a global minimum. Note that there are two conditions: continuity and bounded interval. If either of them is violated, then the theorem is wrong. This theorem shows once more the importance of the property "continuity".

12.1 Practice makes progress

The material is covered in Section 6.1 of the book (5.1 in the first edition). Suggested exercises are

- second edition: **6.1:** 3–5, 6–21, 22–43, 44, 45
- second edition: **6.1:** 63–68, 81, 82 (Applications)
- first edition: **5.1:** 3–39
- first edition: **5.1:** 47–50 (Applications)

Question 1: Find all local and global maxima and minima of the function $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{15}{8}x^2 + \frac{2}{3}$ on the interval $[-3, 3]$.

Question 2: Find the global maximum and the global minimum of $f(x) = \sin^2(x)$ on the interval $[\frac{\pi}{4}, \pi]$.

Question 3: For each of the following functions f, find the critical points and classify them; that is find out whether they are local minima, local maxima or neither.
(a) $f(x) = -\dfrac{2}{3}x^3 - 7x^2 - 24x + 14$
(b) $f(x) = 3 - |16 - 8x|$.
(c) $f(x) = x^3 - 6x^2 + 12x - 8$.

Question 4: Find the global maximum and minimum of the function $f(x) = \dfrac{x^2 + 2x}{e^x}$ on the interval $x \in [0, 4]$.

Question 5: Use the first and second order derivatives to sketch the graph of the function $f(x) = \frac{2}{x^2} + \frac{3}{x^3}$. You have to find the zeros, critical points, inflection points, intervals where the function is increasing and decreasing, and the vertical and horizontal asymptotes if any.

13 Optimization

Goal: Apply our knowledge of extreme values to find "best" values.

Skills:

- Translating word problems into mathematics;
- Identifying variables, distinguishing variables from parameters;
- Using available information to define a function of one variable whose maximum or minimum is the desired solution;

Tips:

- Almost all optimization questions include some kind of trade-off. Sometimes, this trade-off is explict and quite obvious, sometimes it is hidden. Uncovering this trade-off is always helpful in finding and interpreting the result.
- Draw a picture to help you identify the variables and the relationships. Name the variables; write down equations to express their relationships.

13.1 Practice makes progress

I strongy recommend that you read through section 6.2 in the second edition of the book. It contains three detailed optimization problems and lots of detailed practice problems for those three. In the first edition, the corresponding material is scattered a little, but there is some at the end of 5.1 For general practice problens, please go to the previous section and look at the practice problems annotated 'Applications'.

Question 1: A company harvests fish at some rate $h \geq 0$. The yield is $Y(h) = h(500 - h)$ tons of fish, the selling price is \$200 per ton. The cost for harvesting at rate h is $C(h) = 1000h(1 + 0.1h)$ in dollars.
(a) Find the expression of the profit P (= revenue - cost) as a function of harvesting rate.
(b) Find the harvesting rate that maximizes profit.
(c) Find the maximum profit.

Question 2: Find the point on the curve $y = \sqrt{x}$ closest to the point (10,0). Hint: minimize the square of the distance from (10,0) to (x, y).

Question 3: In a movie theatre, the screen on the wall is 20 m high and its base is 10 m above eye level. Let θ denote the viewing angle of the screen, that is, the angle $\angle BET$ from the bottom (B) of the screen to the top (T), measured from the vertex of your eye (E). At what distance x from the screen should you position yourself to maximize θ? (from D. Kouba)

Question 4: The size of a population of bacteria introduced to a nutrient can be described by

$$N(t) = 5000 + \frac{30,000t}{100 + t^2}.$$

Find the maximum size of this population for $t \geq 0$.

Question 5: When a patient takes a drug, the concentration of this drug in the blood first increases fairly quickly and then declines again. A function that describes this behaviour is $y(t) = te^{-t/2}$, where $t \geq 0$ is the time in hours after the drug is taken.
(a) How long after drug administration does the drug concentration reach its maximum value?
(b) What is the maximum concentration?

Question 6: Find the point on the parabola $y = x^2$ that is the closest to the point $(1, 2)$ in the cartesian plane.

Question 7: Consider a population that grows according to the logistic updating function and is harvested at a linear rate $h \geq 0$. The number of individuals of the species satisfies the DTDS $x_{t+1} = x_t(4 - x_t) - hx_t$.
(a) Determine all equilibria of the DTDS.
(b) Determine conditions on h such that all equilibria are biologically relevant.
(c) Assuming your positive equilibrium is stable, determine the value of h that maximizes the yield and state the resulting maximum yield.

Question 8: A golf ball hit with an angle of θ radians and initial velocity of 10m/s will fly for a distance of $d(\theta) = 20.41 \sin(\theta)\cos(\theta)$ metres before it lands (neglecting air resistance). Find the angle θ^* between 0 and $\pi/2$ radians that maximizes the distance flown, and find the maximal distance.

Question 9: The oxygen concentration in a lake over a single day is given by the equation

$$C(t) = 10t^3 - 120t^2 + 210t + 12000,$$

where time, $0 \leq t \leq 24$, is measured in hours. When is the oxygen concentration highest, when is it lowest. What are the maximum and minimum values?

Question 10: When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 80t^2e^{-t}$, where $t \geq 0$ is the time in days and y is the number (in units of thousands of people) of infected people.
(a) At what time will there be the most infected people and how many are there at that time?
(b) When is the number of infected people increasing and when is it increasing?
(c) Identify all points of inflection of the function $y(t)$.
(d) Find any horizontal and vertical asymptotes that may exist.
(e) Draw the graph of the number of infected people as a function of time.

14 L'Hopital's rule

GOAL: Apply derivatives to find indeterminate limits.

Motivation: There are limits we cannot find using algebraic methods, like $\lim_{x\to\infty} \frac{\ln(x)}{x}$. In these cases, it is about who "gets" to infinity "faster"...

Definition: We say that a limit is of indeterminate form of type $\frac{\infty}{\infty}$ or type $\frac{0}{0}$ if it can be written as a fraction

$$\lim_{x\to a} \frac{f(x)}{g(x)},$$

and either $f(x)$ *and* $g(x)$ *both* approach infinity as x approaches a (type $\frac{\infty}{\infty}$) or $f(x)$ *and* $g(x)$ *both* approach zero as x approaches a (type $\frac{0}{0}$).

L'Hopital's theorem tells us how to calculate indeterminate limits without using calculators.

Theorem: If f, g are differentiable and the limit is an indeterminate form of type $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

Other indeterminate forms: Remember, an indeterminate form is any expression where you can seem to logically reason out two completely different answers (because two functions are competing to reach their limit faster). Common ones are:

- product forms: $0 \times \infty$ (or $0 \times -\infty$). Use the identity $ab = \frac{b}{a^{-1}}$ or $ab = \frac{a}{b^{-1}}$ to turn this into a fraction, and an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- difference forms: $\infty - \infty$. If either one is a fraction, you can try putting them over a common denominator. Or, use the rationalization technique: $(a-b) = (a-b) \times \frac{a+b}{a+b} = \frac{a^2-b^2}{a+b}$. Algebraic manipulations will be necessary.
- exponential forms: ∞^0 or 1^∞. Use the identity $f(x)^{g(x)} = e^{g(x)\ln(f(x))}$; then compute the limit of the exponent $g(x)\ln(f(x))$ (which will be a product indeterminate form!) separately, and finally use the continuity of the exponential function to deduce the value of the original limit.

14.1 Practice makes progress

The material is covered in Section 6.4 of the book (5.3 in the first edition). Those sections also talk about "leading behavior", which we don't discuss in this course. Suggested exercises are

- second edition: **6.4:** 17–39
- first edition: **5.3:** 17–39

Question 1: Find the following limits, if they exist, without using sequences of numerical values. If you use L'Hopital's rule, verify first that the limit is of the correct form.

(a) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-2x}\right)$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(x)-x}{x^3}$

(c) $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{\cos(x)-1}$

(d) $\displaystyle\lim_{x\to\infty}(x+3)^{1/x}$

(e) $\displaystyle\lim_{x\to 1}\frac{3(x-1)^2}{e^{2x-2}-x^2}$

(f) $\displaystyle\lim_{x\to 0^+}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]$

(g) $\displaystyle\lim_{x\to 0}\frac{\arctan(x)}{x}$

(h) $\displaystyle\lim_{x\to\pi}\cot^2(x)(x-\pi)^2$

(i) $\displaystyle\lim_{x\to\infty}\frac{e^x-2}{3-2e^x}$

(j) $\displaystyle\lim_{x\to(\pi/2)^+}\frac{\cos(x)}{(x-\pi/2)^2}$

(k) $\displaystyle\lim_{x\to 0}\frac{1-e^x}{1-e^{x/2}}$

(l) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x-1}-\sqrt{x+3}\right)$

(m) $\displaystyle\lim_{x\to 0^+}[x^3\ln(x)]$

(n) $\displaystyle\lim_{x\to\infty}x^4e^{-x}$

(o) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}$

(p) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{\tan(x)}\right)$

(q) $\displaystyle\lim_{x\to 0}x^x$

(r) $\displaystyle\lim_{x\to\infty}\left(1+\frac{y}{x}\right)^x$

Question 2: Find the following limits without using a table of values or a calculator.

(a) $\displaystyle\lim_{x\to\pi/2}\tan(x)\left(x-\frac{\pi}{2}\right)$

(b) $\displaystyle\lim_{x\to\pi/2}\tan(x)\left(x-\frac{\pi}{2}\right)^2$

(c) $\displaystyle\lim_{x\to\pi/2}\tan^2(x)\left(x-\frac{\pi}{2}\right)^2$

(d) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

(e) $\displaystyle\lim_{x\to\infty}(\ln x)^{1/x}$

(f) $\displaystyle\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$

15 Polynomial approximation

GOAL: Use simple functions (polynomials) to approximate complicated functions.

First answer: Tangent line approximation. Given a function $f(x)$ and a (base) point a, we write the tangent line approximation

$$L(x) = f(a) + f'(a)(x - a).$$

Second answer: Let's generalize this idea and find a function whose value, derivative, second derivative, third derivative,... at a point a agree with the corresponding expressions for f.

Definition: The Taylor polynomial of function f of degree n and base point a is given by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In particular, $T_1(x) = L(x)$ is the tangent line. The notation $f^{(n)}$ denotes the n-th derivative of f. The notation $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$ stands for n factorial.

Secant lines and the mean value theorem
With the tangent line, we take value of a function and its derivative at a single point and try to infer something about the value of the function at a nearby point. Using a secant line, we can instead take the value of a function at two points and infer something about the value of its derivative between these two points. This connection is summarized in the Mean Value Theorem.

Mean Value Theorem: If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) then there exists a value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

In other words: there is a point where the slope of the function equals the slope of the line connecting the function values at the endpoints ("average slope").

Rolle's theorem: A special case of the MVT is Rolle's theorem. If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b), and if $f(a) = f(b)$ then there exists a value $c \in (a, b)$ with $f'(c) = 0$.

Note: Differentiability is essential for these theorems to hold. Think about the absolute value function $f(x) = |x|$ on the interval $[-1, 1]$. The function is continuous everywhere and even differentiable everywhere except for at $x = 0$. Furthermore, $f(1) = f(-1) = 1$, so that Rolle's theorem (if it applied) predicted that there is a point with $f'(c) = 0$. But there is no such point.

15.1 Practice makes progress

The material is covered in Sections 5.7 and 6.3 of the book (4.7 and 5.2 in the first edition). The book covers a lot more about secant lines and about error estimates. We don't go into these topics much. Suggested exercises are

- second edition: **5.7:** 1–7, 8–13 (tangent line only), 14–19, 24–27, 28–33
- second edition: **6.3:** 7–10, 11–14
- first edition: **4.7:** 1–7, 8–13 (tangent line only), 14–19, 24–27, 28–33
- first edition: **5.2:** 7–10, 11–14

Question 1: Consider the function $f(x) = (3x + 5)^{4/3}$.
(a) Find the first, second, and third derivative.
(b) Find the Taylor polynomial T_3 using base point $a = 1$ for the function $f(x)$.
(c) Evaluate the error in the approximation by calculating $|f(0.8) - T_3(0.8)|$ to six decimal places.

Question 2: Consider the function $f(x) = 1/x$.
(a) Use the mean value theorem (MVT) to show that there is a number $c \in [1, 2]$ where the function $f(x) = \frac{1}{x}$ has slope -1/2. Find the value of c.
(b) Now consider the same function $f(x) = \frac{1}{x}$ on the interval $[a, b]$ with $0 < a < b < \infty$. What is the value of the derivative that the MVT guarantees exists? At which point $c \in [a, b]$ does it occur?
(c) Now consider the same function $f(x) = \frac{1}{x}$ on the interval $[-1, 1.]$ Since $f(-1) = -1$ and $f(1) = 1$ there should be a point $c \in [-1, 1]$ such that $f'(c) = \frac{1-(-1)}{1-(-1)} = 1$, according to the MVT. Calculate f' and show that no such c can exist? What is wrong in the previous reasoning?

Question 3: Find the Taylor polynomial of degree 3 with base point 4 to approximate the value of $\sqrt{5}$.

Question 4: Find the Taylor polynomial of degree 4 of the function $f(x) = \ln(x^2)$ with base point $a = 1$.

Question 5: Find the Taylor polynomial of degree 3 with base point 0 of the function $f(x) = \tan(x)$. Use this polynomial to approximate $\tan(0.1)$. Compare with the true value.

Question 6: How can you approximate the value $10^{8.1}$? How good is your approximation?

Question 7: (a) Find the Taylor polynomials of degree 1, 2 and 3 for $f(x) = \cos(2x)+x$ with base point $a = \dfrac{\pi}{6}$. (Note that x must be in radians!) Use these polynomials to approximate $f(0.5)$.

Question 8: (a) Find the Taylor polynomials of degree 1, 2 and 3 for $f(x) = \sin(2x)+x^2$ with base point $a = 0$. (Note that x must be in radians!) Use these polynomials to approximate $f(-0.1)$.

Question 9: Consider the function $f(x) = 1 + \sin(2x - 2)$.
(a) Use the linear approximation of f to estimate the value of $f(0.9)$.
(b) Justify from the graph of f why the approximation of $f(0.9)$ in (a) is below the actual value.
(c) Use a Taylor polynomial of degree 3 to approximate $f(0.9)$.

Question 10: Consider the function $f(x) = x^{4/3}$.
(a) Find the tangent line approximation at $x_0 = 1$.
(b) Find the Taylor polynomial of degree three at $x_0 = 1$.
(c) Use the tangent line to estimate $1.01^{4/3}$.
(d) Draw a graph of the function and explain why the tangent line approximation underestimates the true value.

Question 11: Find the Taylor polynomials of degree 3 and 5 of the function $f(x) = \sin(x)$ around $x_0 = 0$.

Question 12: Consider the function $f(x) = e^{3x}$.
(a) Find the tangent line approximation with base point $a = 0$.
(b) Find the Taylor polynomial of degree three with base point $a = 0$.
(c) Compare the value $f(-0.2)$ with the approximation from the tangent line and from the Taylor polynomial.

Question 13:
(a) Find the linear approximation to $f(x) = e^{2\sin x}$ around $a = \pi$.
(b) Use your answer in part (a) to estimate $e^{2\sin(3)}$.
(c) Find the cubic approximation $T_3(x)$ to $f(x) = e^{2x}$ at $a = 0$.

Question 14:
Use a Taylor polynomial of degree three for $f(x) = x^{1/5}$ to estimate $1.1^{1/5}$ without using a calculator.

16 Stability in nonlinear DTDS and Chaos

GOAL: Analyze nonlinear discrete time dynamical systems.

Recall: A linear DTDS $x_{t+1} = rx_t + c$ with $r \neq 1$ has exactly one fixed point $x^* = c/(1-r)$ and this point is stable when $|r| < 1$.

Motivation: A nonlinear DTDS $x_{t+1} = f(x_t)$ can have several fixed points. We want to find a similar simple criterion to determine the stability of each fixed point without doing cobwebbing.

Idea: Since stability is a local concept (all *nearby* solutions converge to the fixed point) we may replace the function f with its tangent line approximation at the fixed point and check stability for the resulting linear system.

Theorem: Let x^* be a fixed point of the DTDS $x_{t+1} = f(x_t)$, i.e., $x^* = f(x^*)$. Then x^* is stable if $|f'(x^*)| < 1$.

Explanation: Write a solution near the fixed point as $x_t = x^* + y_t$ where $|y_t|$ is small. Then the tangent line approximation gives

$$x_{t+1} = f(x_t) \approx f(x^*) + f'(x^*)(x_t - x^*).$$

At the same time $x^* = f(x^*)$, $x_{t+1} = x^* + y_{t+1}$ and $x_t - x^* = y_t$. So, we get the approximation $y_{t+1} \approx f'(x^*)y_t$. This is a linear DTDS; it fixed point $y^* = 0$ is stable if $|f'(x^*)| < 1$. Now, if $y^* = 0$ is stable, then y_t will converge to zero, which means that x_t will converge to x^*.

Examples:

- The Beverton-Holt model $x_{t+1} = \frac{5x_t}{1+x_t}$ has two fixed points. The larger one is stable, the other unstable.
- The Allee effect model $x_{t+1} = \frac{4x_t^2}{1+x_t^2}$ has three fixed points. Two are stable, one is unstable.
- The model with parameter $x_{t+1} = \frac{ax_t}{1+x_t^2}$ has at most one positive fixed point. What is its stability?
- For which values of r is the positive fixed point of the logistic DTDS $x_{t+1} = rx_t(1 - x_t)$ stable?

Beyond Stability: What happens in the logistic model when r is so large that the fixed point is unstable? Use the file `LogisticDTDS.xls` on the course website and try it out. You will be surprised.

16.1 Practice makes progress

The material of this section is covered in sections 6.7 and 6.8 of the second edition of the book (sections of the first edition). Suggested practice problems from the book are

- second edition: **6.7:** 1–12, 13–18, 31, 32, 37, 38
- second edition: **6.8:** 1–4, 9–12, 13–18, 19–26, 29, 30
- first edition: **5.5:** 5–12, 13–20, Applications: 23–26
- first edition: **5.6:** 1–4, 5–8, 9–16

Question 1: Consider a population that grows according to the logistic updating function and is harvested at a linear rate $h \geq 0$. The number of individuals of the species satisfies the DTDS $x_{t+1} = x_t(4 - x_t) - hx_t$.
(a) Determine all equilibria of the DTDS.
(b) Determine conditions on h such that all equilibria are biologically relevant.
(c) Determine conditions on h such that the positive equilibrium is stable.
(d) Determine conditions on h such that the positive equilibrium is unstable.
(e) Determine the value of h that maximizes the yield and state the resulting maximum yield.

Question 2: Consider a population that grows according to the Beverton-Holt updating function and is harvested according to a linear rate h. The number of individuals of the species satisfies the DTDS

$$x_{t+1} = \frac{4x_t}{1 + x_t} - hx_t, \qquad t = 0, 1, 2, \dots.$$

(a) Find the fixed points of this DTDS. [One point should depend on the harvesting rate.]
(b) For which values of h is there a positive fixed point?
(c) Which harvesting rate maximizes the number of individuals harvested at the fixed point?
(d) Is the fixed point with the value of h from part (c) stable? [If you did not get the answer to part (c), use $h = 0.5$.]

Question 3: The density of fish (i.e. number of fish per cubic metre) in a lake is determined by the discrete-time dynamical system $x_{t+1} = \dfrac{4x_t}{1 + 3x_t^2}$, where t is the time in years since the beginning of the observation. Initially, the density is $x_0 = 0.5$.
(a) What will the density be after three years? (4 decimal places are enough)
(b) What is the updating function $f(x)$?
(c) What are the biologically relevant equilibria?
(d) Use the derivative test to determine the stability properties for each of the two equilibria.

Question 4: Consider the DTDS $x_{t+1} = f(x_t)$ for each of the updating functions below.
(a) Find the equilibrium point(s).
(b) Use the derivative test to evaluate the stability of each equilibrium point.
(c) Starting from $x_0 = 5$, calculate x_1, x_2, x_3.
(d) Sketch the graph of the updating function and use cobwebbing to confirm your calculations.

1. $f(x) = \dfrac{1+x}{1+x^2}$
2. $f(x) = \dfrac{5x}{1+4x^2}$
3. $f(x) = rxe^{-x}$ where r denotes a positive parameter
4. $f(x) = \dfrac{2x}{1+0.1x}$

Question 5: Consider the DTDS $M_{t+1} = M_t e^{r(3-\frac{M_t}{3})}, \quad r > 0$.
(a) Calculate the positive equilibrium.
(b) Determine the values of r for which the positive equilibrium is stable.

Question 6: The number of fish in a lake is determined by the DTDS $x_{t+1} = \dfrac{300x_t}{100 + 0.1x_t}$, where t is the time in years since the beginning of the observation. Initially, there are 500 fish.
(a) How many fish will there be after three years?
(b) Find the inverse of the updating function.
(c) How many fish were there one year before the observation started?
Find the two equilibria x_1^*, x_2^* of this system and determine the stability of each one.

Evaluate yourself: Midterm #2 check-in

1. How am I holding up?

   ☐ Very well ☐ OK ☐ Poorly

2. How am I doing towards achieving my specific goals for MAT1330 (page 28)?

   ☐ Very well ☐ OK ☐ Poorly

3. Have I kept to my action plan? If not, can I change either my plan, or my habits, to get back on track?

4. How do I rate my differential Calculus knowledge?

   ☐ Excellent ☐ OK ☐ Poor

5. How do I rate my readiness to solve problems like:

   (a) computing the limit of a function great OK poor

   (b) calculating the derivative of a function great OK poor

   (c) using rules of logarithms to simplify a function before differentiating great OK poor

   (d) using f' and f'' to sketch the graph of f great OK poor

   (e) identifying the features of the graph of f in Calculus terms great OK poor

   (f) finding the local extrema of a function great OK poor

   (g) finding the absolute extrema of a function great OK poor

   (h) solving a word problem asking for an optimal value great OK poor

   (i) knowing when and how to use l'Hopital's rule great OK poor

   (j) finding a Taylor polynomial to approximate a function great OK poor

   (k) using differentiation and the Stability Theorem to analyse the fixed points of a non-linear DTDS great OK poor

(Answer key on page 69.)

17 Newton's method and the intermediate value theorem

GOAL: Find zeros of functions computationally when an explicit solution is not available.

Motivation: How can we even be sure that a solution exists?

The Intermediate Value Theorem (IVT): Suppose f is a continuous function on $[a, b]$. Then for every number N between $f(a)$ and $f(b)$ there is a value $c \in [a, b]$ such that $f(c) = N$.

Next: How can we find such the solution "c" that the IVT promises exists? First convert your question to one of the form "solve $f(x) = 0$".

First try: the bisection method. Cut your interval in half repeatedly, always choosing the half in which the IVT promises you have a solution. This is like the guessing game you play: "Guess my number between 1 and 100." "Is it 50?" "No, too low." "Is it 75?" "No, too high." etc.

Next try: Newton's method. This uses the intersection of the tangent line with the x-axis to estimate the root of $f(x)$; in fact, we can iterate it. Starting with an initial guess of x_0, iterate the DTDS

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

either until $f(x_n)$ is as close to zero as you wish, or until $|x_n - x_{n+1}|$ is as close to zero as you wish.

Note: Newton's method uses the derivative in the denominator. So, if we are close to a critical point, then Newton's method could have massive problems. Most actual algorithms used in root finding therefore use some modified Newton's method.

17.1 Practice makes progress

The material in this section is covered in chapters 6.3 (IVT) and 6.6 (Newton) in the second edition of the book (sections 5.2 and 5.4, respectively in the first edition). Suggested practice problems are

- second edition: **6.3:** 1–4, 15, 16, 19–22, 23–26, 28–33
- second edition: **6.6:** 1–8, 17, 18, 20, 27, 28, 35
- second edition: **5.2:** 1–6, 21, 22, 25–28, 36, 37
- second edition: **5.4:** 1–8, 17, 18, 25, 26, 33

Question 1: Consider the function $f(x) = e^x - 3x$.
(a) Show that this function has a zero between 0 and 1.
(b) Use Newton's method to find an approximation to this solution. Write down the general formula. Begin with $x_0 = 0$ and calculate x_1, x_2, x_3. Please give 6 decimal places for your calculation.

Question 2: Consider the DTDS $N_{t+1} = f(N_t)$ with updating function as below.
(a) Use the Intermediate Value Theorem to show that there is an equilibrium in the closed interval $[a, b]$.
(b) Rewrite the equation for the equilibrium in the form $g(x) = 0$ for some function g, so that you can look for this equilibrium using Newton's method. We have changed to the variable name x because the DTDS of Newton's method is not related to the DTDS above!
(b) Use Newton's method to solve for the equilibrium up to four iterations when $x_0 = a$. Please give decimal 4 points for your calculation.

1. $N_{t+1} = \frac{1}{1+N_t^3}$ on $[0, 1]$, when $x_0 = 0$
2. $N_{t+1} = \ln(3 - N_t^2)$ on $[0, 1]$, when $x_0 = 0$
3. $N_{t+1} = \cos N_t - \frac{1}{2}$ on $[0, \frac{\pi}{2}]$, when $x_0 = 0$
4. $N_{t+1} = N_t^4 - 1$ on $[1, 2]$, when $x_0 = 1$
5. $N_{t+1} = N_t^3 + N_t - \frac{1}{N_t} + 1$ on $[\frac{1}{2}, 1]$, when $x_0 = \frac{1}{2}$
6. $N_{t+1} = 3 - e^{N_t}$ on $[0, 1]$, when $x_0 = 0$

Question 3: Consider the functions $f(x) = e^{x/3}$ and $g(x) = 2 - x^2/2$.
(a) Show that the functions intersect in the interval $[0, 2]$.
(b) Use Newton's method to calculate the intersection point. Write down the general formula for Newton's method. Then start with $x_1 = 1$ and do three iterations.

Question 4: The goal of this question is to show that the function $f(x) = x^3 + x^2 + 3x + 2$ for $x \in (-\infty, \infty)$ has exactly one zero. We split this question into a few subquestions.
(a) Use the intermediate value theorem to show that there exists (at least) one zero.
(b) Use Rolle's theorem to show that if there are two (or more) zeros, then there is at least one critical point.
(c) Show that the function f does not have a critical point.
(d) Put all your arguments together to show that the function has exactly one zero.

Question 5: Use Newton's method to estimate the solution of the equation $\sin\left(x + \frac{\pi}{2}\right) - \dfrac{x}{2} = 0$ by completing the following steps:
(a) Use the Intermediate Value Theorem to show that there is a solution between 0 and $\frac{\pi}{2}$.
(b) Perform three iterations of Newton's method with the initial value $x_0 = \frac{\pi}{4}$ (use 8 decimal places).

Question 6: Consider the equation $x^4 = 4x^3 + 1$.
(a) Show that this equation has a solution between -1 and 0.
(b) Use Newton's method to find an approximation to this solution. Begin with $x_0 = -1$ and calculate x_1, x_2, x_3. Please give 6 decimal points for your calculation.

Question 7: Apply Newton's method to find the intersection of the curves $y = \ln(x)$ and $y = 2x - 4$. Do only three iterations of Newton's method, beginning with the point $x_0 = 1$.

Question 8:

(a) Use both methods to find the value of $\sqrt{2}$ to three digits by setting $f(x) = x^2 - 2$ and looking for a zero. Start with the interval $[1, 2]$ for the bisection method, use the IVT to show that the zero is in this interval. For Newton's method, start with $x_1 = 1$. Compare how long it takes.

(b) In the previous example (i.e., $f(x) = x^2 - 2$) what happens if you start Newton's method at $x = 0$? What when you start at $x = -1$?

(c) Use Newton's method to find an intersection of the curves $y = \ln(x)$ and $y = 2x-4$. Begin with the point $x_1 = 1$ and calculate three iterations.

(d) Find an approximate solution of the equation $x = \cos(x)$.

18 Antiderivatives

GOAL: Reverse the process of differentiation: find a function whose derivative is a given function.

Definition: A *differential equation* (DE) is an equation for the derivative of a function. A pure-time DE contains the independent variable of sought function but not the function itself, whereas an autonomous DE contains the function but not the independent variable. (We study only pure-time equations here; autonomous equations will appear in MAT 1332.)

Definition: An *antiderivative* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. We write

$$F(x) = \int f(x)\, dx$$

and say "F is the integral of f with respect to x." Function $f(x)$ is called the integrand. If $F(x)$ is an antiderivative of $f(x)$ then $F(x) + c$ is also one. We write

$$\int f(x)\, dx = F(x) + c$$

for the set of all antiderivatives and call it the *indefinite integral*.

- The $\int$ symbol and the symbol "dx" serve as a left and right parenthesis; you must include both.
- After you have found one antiderivative, you must include the constant c to account for all the other antiderivatives.

Examples: Everything we have already learned about differentiation can be reformulated in terms of integration. For example, we already know the indefinite integral of many functions.

$$\int 1\, dx = x + c, \qquad \int \frac{1}{2\sqrt{x}}\, dx = \sqrt{x} + c$$
$$\int \frac{1}{x}\, dx = \ln(x) + c, \qquad \int e^x\, dx = e^x + c$$
$$\int \cos(x)\, dx = \sin(x) + c, \qquad \int \sin(x)\, dx = -\cos(x) + c.$$

The goal now is to turn all the differentiation rules into integration rules.

Rules for antidifferentiation:

Power rule: $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + c$ if $n \neq -1$.

Constant product rule: $\int a f(x)\, dx = a \int f(x)\, dx$ if a is a number.

Sum rule: $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$.

18.1 Practice makes progress

The material in this section is covered in chapters 7.1 and 7.2 in the second edition of the book (sections 6.1. and 6.2, respectively in the first edition). Suggested practice problems are

- second edition: **7.1:** 1–9, 10–13, 16–23, Applications: 34–38, 40, 41
- second edition: **7.2:** 1–6, 7–24, 25–34, Applications: 41–46
- first edition: **6.1:** 1–9, 10–13, 16–23, Applications: 34–39
- first edition: **6.2:** 1–9, 10–13, 16–23, Applications: 34–39

Question 1: Find the following indefinite integrals.

(a) $\int \frac{(1+\sqrt{x})^2}{x^2}\,dx$

(b) $\int \frac{(t+1)^2}{2t^3}\,dt$

(c) $\int (x^{-3/4} - x^{3/4})\,dx$

(d) $\int (x^3 + x^{1/3})\,dx$

(e) $\int \frac{(x+1)^2}{x}\,dx$

(f) $\int \frac{(\sqrt{x}+2)^2}{x^2}\,dx$

(g) $\int \frac{(1-x)^2}{x}\,dx$

(h) $\int \frac{(2-x)^2}{x}\,dx$

(i) $\int \frac{(x+3)^2}{x^2}\,dx$

(j) $\int \left(10x^4 - \frac{2}{x} + \frac{4}{\sqrt[3]{x}} - 1\right)\,dx$

Question 2: Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by $f(t) = 3t^3+1$.

Question 3: Find the following indefinite integrals.

(a) $\int \cos(x) + 2\sin(x)\,dx$

(b) $\int \frac{1}{1+x^2}\,dx$ [Hint: consider an inverse trig function]

(c) $\int e^x - \ln(x)\,dx$

(d) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ [Hint: consider an inverse trig function]

Question 4: Verify the following by differentiating the right hand side. Can you spot the mistake that made these answers look plausible (but wrong)? Can you find the correct antiderivative?

(a) $\int \sqrt{\cos^2(x) + \sin^2(x)}\,dx \neq \sin(x) - \cos(x) + c$

(b) $\int x(2x+1)\,dx \neq \frac{1}{2}x^2(x^2+x) + c$

(c) $\int \ln(x^5)\,dx \neq \frac{1}{x^5} \cdot \frac{1}{6}x^6 + c$

19 Integration by substitution

GOAL: Learn to integrate (certain) products.

Motivation: Find the integration rule that corresponds to the chain rule for differentiation: the "anti-chain rule".

Rule:

$$\int f'(g(x))g'(x)dx = f(g(x)) + c$$

Recipe:

1. Define new variable as function of old: $u = g(x)$. (Usually choose the innermost function.)
2. Differentiate the new variable with respect to the old: $\frac{du}{dx} = g'(x)$.
3. "Multiply" by dx to get $du = g'(x)\,dx$.
4. Write the entire integral in terms of the new variable: $\int f'(g(x))g'(x)dx = \int f'(u)du$.
5. Integrate: $\int f'(u)\,du = f(u) + c$.
6. Substitute back: $f(u) + c = f(g(x)) + c$.
7. Check your calculation by differentiating your answer!

Note: The chain rule is a very common differentiation step, so substitution is your first choice after simplifying and checking if you can use one of the previous antidifferentiation rules.

Good candidates for substitution: integrands where you see both a function and its derivative; integrands where the most complicated piece is a composition of functions (try the innermost function); linear substitutions ($u = ax + b$ for some numbers a, b).

Examples:

- $\int e^{3x}\,dx$
- $\int \frac{1}{1+3x}\,dx$
- $\int \cos(2\pi(x-1))\,dx$
- $\int \cos(x)e^{sin(x)}\,dx$
- $\int \frac{e^{-3t}}{(1+e^{-3t})^3}\,dt$
- $\int \frac{\ln(x)}{x}\,dx$
- $\int \frac{(4t+2)^2}{t^2}\,dt$

19.1 Practice makes progress

The material in this section is covered in chapter 7.5 in the second edition of the book (section 6.5 in the first edition). Suggested practice problems are

- second edition: **7.5:** 1–22, 23–35 (evaluate the indefinite integrals only, don't worry about the boundaries of integration), Applications: 80–83, 86, 87
- first edition: **6.5:** 1–22, 23–35 (evaluate the indefinite integrals only, don't worry about the boundaries of integration), Applications: 52–55, 58, 59

Question 1: Find the following indefinite integrals.

(a) $\displaystyle\int \frac{[\ln(x)]^3}{3x}\,dx$

(b) $\displaystyle\int \frac{3x+1}{(3x^2+2x+1)^6}\,dx$

(c) $\displaystyle\int \frac{t}{(1+t^2)^2}\,dt$

(d) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

(e) $\displaystyle\int \frac{(\ln(z))^2}{z}\,dz$

(f) $\displaystyle\int \frac{\sin(\frac{1}{x})}{x^2}\,dx$

Question 2: Find the indefinite integral of each of the following functions. Check your results by differentiating.

(a) $f(x) = (17x - 17)^{17}$

(b) $f(x) = 5^{2x-3}$

(c) $f(x) = \sqrt[3]{12x + 2014}$

(d) $f(x) = \frac{1}{x(\ln x)^2}$

(e) $f(x) = e^{3x}\sqrt{2 - e^{3x}}$

Question 3: Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by

(a) $f(t) = \dfrac{1}{17t + 12}$

(b) $f(t) = 12e^{2t}$

Question 4: Let a be a constant. Differentiate the functions $g(x) = \arctan(ax)$ and $h(x) = \arcsin(ax)$. Now use substitution to find each of the indefinite integral of each of the following functions.

(a) $f(x) = \dfrac{5}{1 + 9x^2}$

(b) $f(x) = \dfrac{5}{\sqrt{1 - 16x^2}}$

(c) $f(x) = \dfrac{5}{16 + 9x^2}$ [Hint: do some algebraic manipulations to make the denominator take the form $1 + a^2x^2$ for some a.]

Question 5: Find the following indefinite integrals

(a) $\displaystyle\int \frac{\tan(x)}{\ln(\cos(x))}\,dx$

(b) $\displaystyle\int \sin(x)e^{\cos(x)}\,dx$

(c) $\displaystyle\int \frac{\cot(x)}{\ln(\sin(x))}\,dx$

(d) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$

(e) $\displaystyle\int \frac{e^x+1}{e^x+x}\,dx$

(f) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

(g) $\displaystyle\int \frac{\cos(x)}{(\sin^2(x))^{1/3}}\,dx$

(h) $\displaystyle\int \cos(x)e^{\sin(x)}\,dx$

(i) $\displaystyle\int \frac{e^x+2}{e^x+2x}\,dx$

(j) $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx$

(k) $\displaystyle\int \left(\frac{2}{x(1+\ln(x))}\right)\,dx$

(l) $\displaystyle\int \frac{\cos(x)}{\sqrt{\sin(x)}}\,dx$

Question 6: Find the anti-derivative $F(x)$ of $f(x) = \dfrac{e^{\arcsin(x)}}{\sqrt{1-x^2}}$ such that $F(0) = 1$.

20 Integration by parts

GOAL: Learn to integrate (certain) products.

Motivation: Find the integration rule that corresponds to the product rule for differentiation: the "anti-product rule".

Rule:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Recipe:

1. Identify the integrand as a product of two functions, $u(x) \cdot v'(x)$, so that $u(x)$ can be differentiated and $v'(x)$ can be integrated (easily).
2. Write down $u(x), u'(x), v'(x), v(x)$.
3. Apply the formula.
4. Repeat if necessary.
5. Check your calculations.

Note: Products are less common than compositions, so this rule is your third choice (after simplifying, and after substitution).

Good candidates for integration by parts are: integrands that have functions from two different families, multiplied together; integrands with an ugly function that has a nice derivative. Remember that, like substitution, integration by parts just swaps one integral for another; to solve the new integral, you may need to use substitution or integration by parts again!

Examples:

- $\int \ln(x)dx$
- $\int x^2e^{3x}dx$
- $\int x\ln(x)dx$
- $\int \frac{\ln(x)}{x^2}dx$
- $\int e^x \sin(x)dx$ (keep your eyes open for this one!)

20.1 Practice makes progress

The material in this section is covered in chapter 7.5 in the second edition of the book (section 6.5 in the first edition). Suggested practice problems are

- second edition: **7.5:** 36–47, Applications: 50, 51, 56, 57
- first edition: **6.5:** 36–47, Applications: 84, 85

Question 1: Find the following indefinite integrals.

(a) $\int (x+1)\sin(x)\,dx$

(b) $\int x^2\cos(x)\,dx$

(c) $\int 16x^3\ln(7x)dx$

Question 2: Find the indefinite integral of each of the following functions. Check your results by differentiating.

(a) $f(x) = \frac{x}{2}\cos(5x)$

(b) $f(x) = \sqrt{x}\ln x$

(c) $f(x) = \arcsin(x)$

(d) $f(x) = x3^x$

(e) $f(x) = x^2e^{-x}$

Question 3: Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by

(a) $f(t) = (t+t^2)e^{-t}$

(b) $f(t) = 3t\cos(t^2)$

Question 4: Find the following indefinite integrals

(a) $\int x\ln(x)\,dx$

(b) $\int (x+1)\sin(x)\,dx$

(c) $\int (x+1)\cos(x)\,dx$

(d) $\int (x+1)\ln(x)\,dx$

(e) $\int (x-2)\sin(x)\,dx$

(f) $\int \sqrt[3]{x}\ln(x)\,dx$

(g) $\int 3x^2\cos(0.5x)\,dx$

Question 5: Find the function $f(x)$, such that $f''(x) = \ln(x)$ and $f(1) = f'(1) = 0$.

Question 6: Let $V(t)$ be the volume of a benign tumour in cm^3 after t years. For $t \geqslant 0$, suppose that $V(t)$ satisfies the following differential equation

$$\frac{dV}{dt} = (1+t)e^{-t}.$$

a) If initially $V(0) = 1$, find $V(t)$.

b) Compute $\lim_{t\to\infty} V(t)$ and interpret.

c) Use Newton's method to find when the volume of the tumour will be $2\ \text{cm}^3$. Use 5 decimal places in your computations and find the answer with 3 decimal places of precision.

21 Definite integrals and the fundamental theorem of calculus

GOAL: To relate the area problem to the problem of finding anti-derivatives.

Skill: Summation notation is a mathematical shorthand for a sum of terms in a pattern. It's a more condensed way to write an expression where you might use $\cdots$ to skip intermediate terms. For example,

$$\sum_{i=3}^{n} x_i = x_3 + x_4 + x_5 + \cdots + x_n.$$

Definition: A Riemann sum of $f(x)$ on the interval $[a, b]$ is an expression of the form

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

where

- n is a positive integer, representing the number of subintervals into which you've divided $[a, b]$;
- $\Delta x = (a - b)/n$ is the width of each subinterval;
- x_i^* is a point in the ith subinterval (example: the left endpoint, the right endpoint, the midpoint, etc).

Note: A Riemann sum S_n is an estimate of the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

Definition: The definite integral of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} S_n.$$

Fundamental Theorem of Calculus (one part) : Evaluation Theorem If f is continuous on an interval $[a, b]$ then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

Note: If the function is not above the x-axis on $[a, b]$, then the definite integral can be negative. The definite integral measure the net area above the x-axis = area above the x-axis minus the area below the x-axis. See MAT1332.

21.1 Practice makes progress

The material in this section is covered in Sections 7.3 and 7.4 of the second edition of the book (guess: section 6.3, 6.4 in the first edition). What Adler and Lovric call the Fundamental Theorem of Calculus Part I is what I call the Fundamental Theorem of Calculus Part II; so be it. Suggested practice problems are

- second edition:
  7.3: 1–15,
  7.4: 5–37, Applications: 66–71,
  7.5: 48–59 (now doing the definite integrals as well).

Question 1: Sketch the graph of $y = f(x)$ between $x = a$ and $x = b$. Write down the Riemann sum with the given number n of intervals for estimating the area under the curve, and check with your graph. Then evaluate the Riemann sum. Can you determine if it is an overestimate or an underestimate?

(a) $f(x) = x^3$, on the interval $[0, 1]$, with $n = 3$ subintervals, using the right endpoint rule.

(b) $f(x) = \sin(x)$, on the interval $[0, \pi/2]$, with $n = 3$ subintervals, using the left endpoint rule.

(c) $f(x) = \cos(x)$, on the interval $[0, \pi/2]$, with $n = 3$ subintervals, using the left endpoint rule.

(d) $f(x) = x^2$, on the interval $[0, 300]$, with $n = 10$ subintervals, using the left endpoint rule.

Question 2: Determine the following definite integrals by first finding an anti-derivative and then using the Fundamental Theorem of Calculus. Then sketch the graph and decide if your answer seems reasonable.

(a) $\displaystyle\int_0^{\pi} \sin(x)\, dx$

(b) $\displaystyle\int_1^{7} \frac{\ln(x)}{x}\, dx$

(c) $\displaystyle\int_0^{4} x \ln(5 - x)\, dx$

Evaluate yourself: Final exam check-in

1. How do I rate my knowledge and skills in:

   (a) algebra, functions and trigonometry great OK poor

   (b) DTDS great OK poor

   (c) differential Calculus great OK poor

   (d) integral Calculus great OK poor

2. How do I rate my readiness to solve problems on the exam like:

   (a) simplifying an algebraic or logarithmic expression great OK poor

   (b) setting up, evaluating, and drawing conclusions about a DTDS great OK poor

   (c) differentiating a function great OK poor

   (d) sketching the graph of a function using Calculus great OK poor

   (e) optimizing a function using Calculus, given constraints great OK poor

   (f) knowing when and how to use l'Hopital's rule great OK poor

   (g) approximating functions using Taylor polynomials great OK poor

   (h) determining the stability of a fixed point of a DTDS using Calculus great OK poor

   (i) using Newton's method to approximate solutions when exact answers can't be found great OK poor

   (j) identifying a list of methods to determine an antiderivative of a function great OK poor

   (k) finding the indefinite integral of a function, using various methods great OK poor

   (l) computing a definite integral using the Evaluation Theorem great OK poor

3. What will I do to prepare for the final exam?

(No answer key — you've got this. Good luck!)

Evaluate yourself: Answer keys

About Self-Regulated Learning A set of modules entitled *Goals and Growth* was developed by Dr. Alison Flynn, University Chair in Teaching at uOttawa, as part of a project on Self-Regulated Learning (SRL) and achieving a Growth Mindset for post-secondary students. In brief, by helping you to think about your learning process (what, when, how, why and even where) you become much more effective at taking control of your education and life. (You also start to recognize which of your habits help or hinder your learning.)

These "Evaluate yourself" pages are a sampling of activities in the same theme as the module.

Calculus readiness (page 11):

1. Your self-awareness about your learning is the key to learning more.
2. Thinking in greater detail about your strengths and weaknesses relative to the learning objectives is a big part of self-regulated learning — at the first step in forming an action plan.
3. Math is about practice, and about solving problems.
   - Being more or less prepared than your friends is not an objective measure.
   - Your high school grades don't tell the whole story: most of this material is from Grades 10 and 11, and you may not have been assessed on it since.
   - Understanding a lecture is like appreciating a piece of music — that is to say, not all that related to being able to reproduce it on your own.
   - Working with friends is EXCELLENT — but remember that figuring out how to start a problem, and finish it correctly on your own, is the ultimate and most important measure of your understanding.
4. Whether you have a *fixed mindset* (for example, believing that math ability is just something you are born with or not) or a *growth mindset* (for example, believing that by putting in the right effort you will learn) has a substantial impact on your learning — how motivated you are to try, how you view your successes and your failures, and how well you tackle the next challenge.
5. Choose a strategy! (Well, not avoidance, and not just redoing the same thing over and over again.) Remember that getting the right answer is the reward, not the only goal. Learn how to do problems, using all the resources available, and test yourself by then tackling some on your own.

DTDS (page 21):

1. A DTDS is a discrete-time dynamical system. Mathematically, it is a choice of discrete time interval t, and an updating function f that expresses the time-evolution of the quantity x_t, via the formula $x_{t+1} = f(x_t)$ for each $t \in \mathbb{N}$. A DTDS describes how a complex system evolves over time, under the hypothesis that its evolution depends on the state it is in.

2. The updating function is part of the DTDS. The updating function is just a function; it could be any function. The DTDS is a description of a dynamical system, and includes also the time step and perhaps the initial condition.

3. A cobweb diagram is a tool to geometrically produce estimates of the values $x_1, x_2, x_3, \cdots$ quickly and efficiently, without using a calculator. Although it doesn't give accurate values, it shows you what will happen in the long run (much better than calculating values would do).

4. The cobweb is sketched on a graph of x_{t+1} versus x_t; "time" is a video rolling as you draw the cobweb, and at each right turn you have the next value of x_t. The solution is a set of points on a graph of x_t versus t: at each time t, the dot represents the value of x_t.

5. Well, you could invent a DTDS that doesn't work, like $x_{t+1} = \sqrt{x_t}$ and $x_0 = -1$. But in any normal context (and this course), the solution is the description of what happens over time (the set of values $x_0, x_1, x_2, \ldots$) which always exists (it just might be hard to compute or predict).

6. A linear DTDS has a nice solution formula, which is great. It is rare to find a general solution formula for a nonlinear DTDS (and beyond the scope of this course).

7. A linear DTDS has a linear updating function. A nonlinear DTDS has a nonlinear updating functions.

8. Choosing the updating function of a DTDS is the key step in this kind of mathematical modeling; it relies on understanding the biological process and is verified using data. In this course, you are (only) expected to become proficient at choosing the updating function for a *linear* DTDS based on the given data.

9. No difference, same concept. Favourite notation: x^* (a "decorated" x).

10. Algebraically: if the updating function is $y = f(x)$ then solve $x = f(x)$ for x. Graphically: find where the graph of $y = f(x)$ crosses the graph of $y = x$. Special case: for a linear DTDS, there's a formula.

11. Graphically: start a cobweb on each side of the fixed point and see where it goes. Special case: for a linear DTDS, there's a criterion involving the slope of the updating function.

12. Systems evolve to a stable steady state, when there is one. Mathematically, a system could be an an unstable steady state, and stay there forever, but in real life, small perturbations mean the system is soon just in a nearby state, and then evolving away.

Midterm #1 Check-in (page 28):

1. (I have my own goals. Some of them I'll never say out loud and it's OK if you don't either.)

2. Some common answers: "thirst for knowledge"; "necessary knowledge to succeed in my program"; "a rite of passage"; "I don't know". If you don't know, and are finding it hard to be motivated, that could be the problem — come to office hours, or the Science mentoring center, or the Science Undergraduate Office, to chat!

3. The "process" ones are the ones that endure the most...

4. Be honest, but not too hard on yourself.

5. Good action items include being proactive: scheduling time to work fresh problems, to go to office hours or the help centre as needed, to review the material that will be on the midterm.

Midterm #2 Check-in (page 52):

1. Self-regulated learning isn't just about studies; it's about assessing yourself and taking control of your situation. If things feel out of control, it's time to reach out and tap into the resources here (Science Buddies, Science Undergraduate Office, SASS, your friends, your doctor,...) so you can achieve your potential in the long run.

2. As far as math goes, your goal is to do problems (MapleTA, these notes, the textbook) and your resources include: attending class, participating in the DGD, taking part in study groups, coming to office hours, going to the math help center.

3. Hopefully things are going well enough. If not: drop date is approaching. It's time to commit yourself to an action plan that will enable your success; talk to your academic advisor as needed.

4. Now to the task at hand!

5. Practice, practice, practice.