MATH 1102 Fall & Winter 2017/2018

Assignment 10 Solutions

The following questions are to be turned in to be graded.

1. Consider $\mathbf{v} = (2,3,4) \in \mathbb{R}^3$ (over $\mathbb{R}$). In each part below, we give an ordered basis of $\mathbb{R}^3$ (you do *not* have to prove that they are bases here). In parts (a) to (c), you are asked to find coordinate vectors. If you can find a coordinate vector by inspection, that is fine. Just briefly show that it works.

   (a) (1 mark) If $B_1 = \{(1,1,1),(0,1,1),(0,0,1)\}$, then find $[\mathbf{v}]_{B_1}$.

   (b) (0.5 marks) If $B_2 = \{(0,0,1),(0,1,1),(1,1,1)\}$, then find $[\mathbf{v}]_{B_2}$.

   (c) (1 mark) If $B_3 = \{(1,0,1),(1,1,1),(1,2,3)\}$, then find $[\mathbf{v}]_{B_3}$

   (d) (0.5 marks) If $[\mathbf{u}]_{B_3} = (3,-1,2)$ with B_3 as in part (c), find $\mathbf{u}$.

   Solution:

   (a) We must find c_1, c_2, c_3 with

   $$c_1(1,1,1) + c_2(0,1,1) + c_3(0,0,1) = (2,3,4).$$

   Due to the presence of many zeroes in the basis vectors, we can find these by inspection (since we are taking a linear combination of basis vectors, we know there is a unique solutions). Multiplying through, the first components yield $c_1 = 2$. The second components yield $c_1 + c_2 = 3$, and so $c_2 = 3 - c_1 = 3 - 2 = 1$. The third components yield $c_1 + c_2 + c_3 = 4$, and so $c_3 = 4 - c_1 - c_2 = 4 - 2 - 1 = 1$. We have

   $$2(1,1,1) + (0,1,1) + (0,0,1) = (2,3,4).$$

   Thus $[\mathbf{v}]_{B_1} = (c_1, c_2, c_3) = (2,1,1)$.

   (b) B_2 consists of the same vectors of B_1 in the reverse order. We can reuse our work from part (a). We have

   $$(0,0,1) + (0,1,1) + 2(1,1,1) = (2,3,4),$$

   and so $[\mathbf{v}]_{B_2} = (1,1,2)$.

   (c) We must find c_1, c_2, c_3 with

   $$c_1(1,0,1) + c_2(1,1,1) + c_3(1,2,3) = (2,3,4).$$

   We form the matrix with these vectors as columns and row reduce (details of the row reduction are omitted).

   $$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Since row operations preserve dependence relations among columns, we can see that

$$0(1,0,1)+(1,1,1)+(1,2,3)=(2,3,4).$$

(Note that this could have likely been found by inspection). Thus $[\mathbf{v}]_{B_3} = (0,1,1)$.

(d) We have $\mathbf{u} = 3(1,0,1) - 1(1,1,1) + 2(1,2,3) = (4,3,8)$.

2. Let V be an finite dimensional vector space over F with ordered basis B.

   (a) (1 mark) Prove $[\mathbf{0}]_B = \mathbf{0}$ (note that the zero vector on the left side is the zero vector in V, while the zero vector on the right side is the zero vector in F^n).

   (b) (3 marks) Prove that if $[\mathbf{v}_1]_B, ..., [\mathbf{v}_k]_B$ are linearly independent in F^n, then $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent in V.

Solution:

(a) Let $B = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$. Then $\mathbf{0} = 0\mathbf{v}_1 + ... + 0\mathbf{v}_n$, and so $[\mathbf{v}]_B = (0, ..., 0) = \mathbf{0}$.

(b) Suppose $[\mathbf{v}_1]_B, ..., [\mathbf{v}_k]_B$ are linearly independent. Suppose

$$c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}.$$

We then have

$$\begin{aligned} \mathbf{0} &= [\mathbf{0}]_B \\ &= [c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n]_B \\ &= [c_1\mathbf{v}_1]_B + ... + [c_n\mathbf{v}_n]_B && \text{(by theorem from class)} \\ &= c_1[\mathbf{v}_1]_B + ... + c_n[\mathbf{v}_n]_B && \text{(by theorem from class).} \end{aligned}$$

Since the $[\mathbf{v}_i]_B$ are linearly independent, this implies that $c_1 = ... = c_n = 0$. Hence $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent.

3. (3 marks) In this question, consider $P_2(\mathbb{R})$ with ordered basis $B = \{x^2, x, 1\}$.

   (a) Write down the coordinate vectors (with respect to B) for the polynomials x^2+2x+1, x^2+x+2, and x^2+4x-1. Form the matrix with columns equal to these coordinate vectors and put it in reduced row echelon form. Are the polynomials linearly dependent or independent? If they are linearly dependent, write down an explicit dependence relation.

   (b) Write down the coordinate vectors (with respect to B) for the polynomials x^2+2x+1, x^2+x+2, and x^2+4x+1 (only the constant term in the last polynomial has been changed from part (a)). Form the matrix with columns equal to these coordinate vectors and put it in reduced row echelon form. Are the polynomials linearly dependent or independent? If they are linearly dependent, write down an explicit dependence relation.

Solution:

(a) We have

$$[x^2+2x+1]_B = (1,2,1), [x^2+x+2]_B = (1,1,2), \text{ and} [x^2+4x-1]_B = (1,4,-1).$$

Writing the coordinate vectors as columns in a matrix and row reducing yields

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since row operations preserve dependence relations among the columns of a matrix, we see that $3(1,2,1) - 2(1,1,2) = (1,4,-1)$, and so

$$3(1,2,1) - 2(1,1,2) - (1,4,-1) = (0,0,0).$$

Hence the same is true for the polynomials:

$$3(x^2+2x+1) - 2(x^2+x+2) - (x^2+4x-1) = 0.$$

The polynomials are linearly dependent.

(b) We have

$$[x^2+2x+1]_B = (1,2,1), [x^2+x+2]_B = (1,1,2), \text{ and} [x^2+4x-1]_B = (1,4,1).$$

Writing the coordinate vectors as columns in a matrix and row reducing yields

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the RREF is the identity matrix, the columns of the original matrix are independent. Thus the polynomials are also independent.

4. (1 mark) Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x,y) = (2x-3y, -5x+y, 7x-y)$. Find a matrix A such that

$$T\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}.$$

Then use a theorem from class about matrix mappings to deduce that T is a linear map.

Solution: We have

$$\begin{aligned} T\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2x-3y \\ -5x+y \\ 7x-y \end{bmatrix} \\ &= \begin{bmatrix} 2x \\ -5x \\ 7x \end{bmatrix} + \begin{bmatrix} -3y \\ y \\ -y \end{bmatrix} \\ &= x\begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix} + y\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 2 & -3 \\ -5 & 1 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In class we proved that a map given by $T(\mathbf{v}) = A\mathbf{v}$ (for a given matrix A) is linear. Thus T is linear.

5. Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(ax^2 + bx + c) = a + b + c$.

   (a) (0.5 marks) Find $T(4x^2 - 7x + 9)$.

   (b) (2 marks) Show that T is a linear map.

   (c) (0.5 marks) Describe the set $V = \{\mathbf{v} \in P_2(\mathbb{R}) : T(\mathbf{v}) = \mathbf{0}\}$. This set should be familiar to you. Is V a subspace of $P_2(\mathbb{R})$? You may refer to any previously seen work, such as material on the midyear test, to answer this question.

   Solution:

   (a) $T(4x^2 - 7x + 9) = 4 + (-7) + 9 = 6$.

   (b) Let $p(x) = a_1x^2 + b_1x + c_1$ and $q(x) = a_2x^2 + b_2x + c_2$ be elements of $P_2(\mathbb{R})$, and let $\alpha \in \mathbb{R}$. Then

   $$\begin{aligned} T(p(x) + q(x)) &= T\big((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)\big) \\ &= (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) \\ &= (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) \\ &= T(p(x)) + T(q(x)) \end{aligned}$$

   and

   $$\begin{aligned} T(\alpha p(x)) &= T\big((\alpha a_1)x^2 + (\alpha b_1)x + (\alpha c_1)\big) \\ &= \alpha a_1 + \alpha b_1 + \alpha c_1 \\ &= \alpha(a_1 + b_1 + c_1) \\ &= \alpha T(p(x)). \end{aligned}$$

   Thus T is linear.

   (c) Taking $\mathbf{v} = ax^2 + bx + c$, we have $T(\mathbf{v}) = \mathbf{0}$ if and only if $a + b + c = 0$. Thus

   $$V = \{ax^2 + bx + c \in P_2(\mathbb{R}) : a + b + c = 0\}.$$

   On the midyear test, it was shown that this is a subspace of $P_2(\mathbb{R})$.

6. In each part below, we give a function $T : P(\mathbb{R}) \to P(\mathbb{R})$.

   (a) (1.5 marks) Define T by $T(p(x)) = xp(x)$. Show that T is a linear map.

   (b) (1 mark) Define T by $T(p(x)) = (p(x))^2$. Show that T is not a linear map.

   (c) (1.5 marks) Define T by $T(p(x)) = p(x^2)$. Show that T is a linear map. Hint: note that if $p(x) = \sum_{i=0}^{n} a_i x^i$, then $T(p(x)) = \sum_{i=0}^{n} a_i x^{2i}$.

Solution: In each part below, let $p(x), q(x) \in P(\mathbb{R})$ and let $c \in \mathbb{R}$.

(a) We have

$$\begin{aligned} T(p(x)+q(x)) &= x(p(x)+q(x)) \\ &= xp(x)+xq(x) \\ &= T(p(x))+T(q(x)) \end{aligned}$$

and

$$\begin{aligned} T(cp(x)) &= x(cp(x)) \\ &= c(xp(x)) \\ &= cT(p(x)). \end{aligned}$$

Thus T is linear.

(b) We use $p(x) = x$ and $q(x) = 1$ for our counterexample. We have $T(x) = x^2$ and $T(1) = 1$, and so $T(x)+T(1) = x^2+1$. However, $T(x+1) = (x+1)^2 = x^2+2x+1 \neq T(x)+T(1)$. Thus T is not linear. Any single counterexample to either of the linear map properties is sufficient.

(c) Note that p and q may not have the same degree. Let n be the largest of the two degrees. Then we can write $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{n} b_i x^i$, where we use a coefficient of 0 for any terms that do not appear in the polynomial. We have

$$\begin{aligned} T(p(x)+q(x)) &= T\left(\sum_{i=0}^{n}(a_i+b_i)x^i\right) \\ &= \sum_{i=0}^{n}(a_i+b_i)x^{2i} \qquad \text{(def. of } T\text{)} \\ &= \sum_{i=0}^{n} a_i x^{2i} + \sum_{i=0}^{n} b_i x^{2i} \qquad \text{(sums are finite)} \\ &= T\left(\sum_{i=0}^{n} a_i x^i\right) + T\left(\sum_{i=0}^{n} b_i x^i\right) \qquad \text{(def. of } T\text{)} \\ &= T(p(x))+T(q(x)) \end{aligned}$$

and

$$\begin{aligned} T(cp(x)) &= T\left(\sum_{i=0}^{n}(ca_i)x^i\right) \\ &= \sum_{i=0}^{n}(ca_i)x^{2i} \qquad \text{(def. of } T\text{)} \\ &= c\sum_{i=0}^{n} a_i x^{2i} \end{aligned}$$

$$\begin{aligned} &= cT\left(\sum_{i=0}^{n} a_i x^i\right) \qquad \text{(def. of } T\text{)} \\ &= cT(p(x)). \end{aligned}$$

Thus T is linear.

7. Recall that we proved that a linear map $T : V \to W$ is completely determined by where it sends the elements of a basis of V.

   (a) (1 mark) Suppose $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map that satisfies $T_1(1,0) = (2,-3)$ and $T_1(0,1) = (-1,2)$. Find an explicit formula for $T_1(x,y)$.

   (b) (1 mark) Suppose $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map that satisfies $T_2(1,0) = (1,1)$ and $T_2(0,1) = (1,1)$. Find an explicit formula for $T_2(x,y)$. Note that T_2 sends both basis vectors to $(1,1)$. Does this mean that T_2 sends all elements of $\mathbb{R}^2$ to $(1,1)$?

   (c) (2 marks) Suppose $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map that satisfies $T_3(1,1) = (2,-3)$ and $T_3(1,-1) = (-1,5)$. Find an explicit formula for $T_3(x,y)$. Hint: one way to proceed is to find c_1 and c_2 with $(x,y) = c_1(1,1) + c_2(1,-1)$ (c_1 and c_2 will depend on x and y), and then use the fact that T is linear.

   Solution:

   (a) We have $(x,y) = x(1,0) + y(0,1)$. Thus

$$\begin{aligned} T_1(x,y) &= T_1\left(x(1,0) + y(0,1)\right) \\ &= T_1\left(x(1,0)\right) + T_1\left(y(0,1)\right) \qquad (T \text{ is linear}) \\ &= xT_1(1,0) + yT_1(0,1) \qquad (T \text{ is linear}) \\ &= x(2,-3) + y(-1,2) \\ &= (2x - y, -3x + 2y). \end{aligned}$$

   (b) We have $(x,y) = x(1,0) + y(0,1)$. Thus

$$\begin{aligned} T_2(x,y) &= T_2\left(x(1,0) + y(0,1)\right) \\ &= xT_2(1,0) + yT_2(0,1) \qquad (T \text{ is linear}) \\ &= x(1,1) + y(1,1) \\ &= (x + y, x + y). \end{aligned}$$

   No, T_2 does not send all elements of $\mathbb{R}^2$ to $(1,1)$. For example, $T_2(2,3) = (5,5)$. It does send everything to a scalar multiple of $(1,1)$.

   (c) We must find $c_1, c_2 \in \mathbb{R}$ such that $c_1(1,1) + c_2(1,-1) = (x,y)$. Writing these vectors as columns and row reducing, we have

$$\begin{bmatrix} 1 & 1 & x \\ 1 & -1 & y \end{bmatrix} \to \begin{bmatrix} 1 & 1 & x \\ 0 & -2 & y - x \end{bmatrix} \; R2 - R1$$

$$= \begin{bmatrix} 1 & 1 & x \\ 0 & 1 & \frac{x-y}{2} \end{bmatrix} \quad (-1/2)R2$$
$$= \begin{bmatrix} 1 & 0 & \frac{x+y}{2} \\ 0 & 1 & \frac{x-y}{2} \end{bmatrix} \quad R1 - R2.$$

Thus
$$\frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1) = (x,y).$$

We have
$$\begin{aligned} T_3(x,y) &= T\left(\frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)\right) \\ &= \frac{x+y}{2}T(1,1) + \frac{x-y}{2}T(1,-1) \qquad (T \text{ is linear}) \\ &= \frac{x+y}{2}(2,-3) + \frac{x-y}{2}T(-1,5) \\ &= \left(\frac{x+3y}{2}, x-4y\right). \end{aligned}$$

8. Let V and W be vector spaces over the same field F. This question shows that the two conditions for a linear map may be replaced with a single condition.

   (a) (2 marks) Suppose $T : V \to W$ is a function such that for any $c \in F$ and any $\mathbf{u}, \mathbf{v} \in V$, we have
   $$T(c\mathbf{u} + \mathbf{v}) = cT(\mathbf{u}) + T(\mathbf{v}).$$
   Prove that T is a linear map.

   (b) (1 mark) Suppose $T : V \to W$ is a linear map. Prove that for any $c \in F$ and any $\mathbf{u}, \mathbf{v} \in V$, we have
   $$T(c\mathbf{u} + \mathbf{v}) = cT(\mathbf{u}) + T(\mathbf{v}).$$

Solution:

(a) Suppose that for any $c \in F$ and any $\mathbf{u}, \mathbf{v} \in V$, we have $T(c\mathbf{u} + \mathbf{v}) = cT(\mathbf{u}) + T(\mathbf{v})$. Taking $c = 1$ yields
$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}).$$
Taking $\mathbf{v} = \mathbf{0}$ in the original assumption yields
$$T(c\mathbf{u}) = cT(\mathbf{u}) + T(\mathbf{0}).$$
To complete the proof, we must show that $T(\mathbf{0}) = \mathbf{0}$ (note that if T is linear, then this is true. However, we can not assume that T is linear here since that is what we are trying to prove!). We go back to the original assumption with $c = -1$ and $\mathbf{u} = \mathbf{v}$. Then $c\mathbf{u} + \mathbf{v} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$. Thus
$$T(\mathbf{0}) = (-1)T(\mathbf{u}) + T(\mathbf{u}) = \mathbf{0}.$$
Hence it follows that $T(c\mathbf{u}) = cT(\mathbf{u})$, and so T is linear.

(b) Suppose T is linear. Then

$$\begin{aligned} T(c\mathbf{u}+\mathbf{v}) &= T(c\mathbf{u})+T(\mathbf{v}) && \text{(since } T \text{ is linear)} \\ &= cT(\mathbf{u})+T(\mathbf{v}) && \text{(since } T \text{ is linear).} \end{aligned}$$

The following are *suggested exercises* and are *not* be to turned in.

(i) Let F be a field, and consider the vector space F^n over F. Let $\mathbf{e}_i \in F^n$ be the vector with ith entry equal to 1 and all other entries equal to 0. Then $B = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is the standard basis of F^n (over F). Let $\mathbf{v} \in F^n$. Prove that $[\mathbf{v}]_B = \mathbf{v}$.

Solution: Let $\mathbf{v} = (c_1, ..., c_n)$ be a vector in F^n. Then $\mathbf{v} = c_1\mathbf{e}_1 + ... + c_n\mathbf{e}_n$ and so $[\mathbf{v}]_B = (c_1, ..., c_n) = \mathbf{v}$.

(ii) Consider the vector space $M_{22}(\mathbb{R})$ over $\mathbb{R}$. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$.

(a) Find $[A]_{B_1}$, where B_1 is the ordered basis

$$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

(b) Find $[A]_{B_2}$, where B_2 is the ordered basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Solution: This problem is also solved on MS-LAP.

(a) We must find $c_1, ..., c_4 \in \mathbb{R}$ with

$$\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

The presence of many zeroes in the basis matrices makes it possible to find this by inspection (since B_1 is a basis, we know that the c_i exist and are unique). Looking at the entries in the first row and first column yields $c_1 = 1$. First row second column yields $-c_1+c_2 = -2$, and so $c_2 = -2+c_1 = -2+1 = -1$. Second row first column yields $-c_1 + c_2 + c_3 = 3$, and so $c_3 = 3+c_1-c_2 = 3+1-(-1) = 5$. Finally, second row second column yields $c_1 + c_2 - c_3 - c_4 = 5$, and so $c_4 = -5+c_1+c_2-c_3 = -5+1-1-5 = -10$. Thus $[A]_{B_1} = (1, -1, 5, -10)$.

(b) It is clear that

$$\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $[A]_{B_2} = (1, -2, 3, 5)$.

(iii) Consider the vector space $V = \text{span}\{e^x, e^{-x}\}$ over $\mathbb{R}$ (this is a subspace of the vector space consisting of all functions mapping from $\mathbb{R}$ to $\mathbb{R}$). The hyperbolic sine and cosine functions are defined as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

One may show that e^x and e^{-x} are linearly independent (you do not have to do this here), and so $B = \{e^x, e^{-x}\}$ is an ordered basis for V. Find the coordinate vectors for $\sinh(x)$ and $\cosh(x)$ with respect to B.

Solution: Since $\sinh(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$, we have $[\sinh(x)]_B = \left(\frac{1}{2}, -\frac{1}{2}\right)$. Since $\cosh(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$, we have $[\cosh(x)]_B = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(iv) Consider the $V = \text{span}\{x^2, 2^x, \sin x, \cos x\}$. This is a subspace (over $\mathbb{R}$) of the vector space of functions mapping from $\mathbb{R}$ to $\mathbb{R}$. The set $B = \{x^2, 2^x, \sin x, \cos x\}$ is an ordered basis of V (you do not have to show this here).

(a) Find the coordinate vector with respect to the basis B for each of the following elements of V:

$$\begin{aligned} \mathbf{v}_1 &= x^2 + \sin x - \cos x \\ \mathbf{v}_2 &= 2^x + 2\sin x + 2\cos x \\ \mathbf{v}_3 &= 2x^2 - 2^x - 4\cos x \\ \mathbf{v}_4 &= x^2 + 2^x + \sin x + \cos x. \end{aligned}$$

(b) Using the coordinate vectors from part (a), determine if the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent or linearly independent. If they are linearly dependent, write down an explicit dependence relation.

(c) Using the coordinate vectors from part (a), determine if the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_4$ are linearly dependent or linearly independent. If they are linearly dependent, write down an explicit dependence relation.

Solution:

(a) We have

$$\begin{aligned} [\mathbf{v}_1]_B &= (1, 0, 1, -1), \\ [\mathbf{v}_2]_B &= (0, 1, 2, 2), \\ [\mathbf{v}_3]_B &= (2, -1, 0, -4), \text{ and} \\ [\mathbf{v}_4]_B &= (1, 1, 1, 1). \end{aligned}$$

(b) We form the matrix with the coordinate vectors as columns and row reduce:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the RREF, we see that the third column is a linear combination of the first two. Thus the same dependence relation holds among the coordinate vectors, and hence among the vectors themselves. Hence $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent with $2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3$.

(c) We form the matrix with the coordinate vectors as columns and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the columns of the RREF are linearly independent, so are the coordinate vectors. Hence $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ are linearly independent.

(v) Let V and W be vector spaces over the same field F. Consider the zero map $O : V \to W$ defined by $O(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. Show that O is a linear map.

Solution: Let $\mathbf{u}, \mathbf{v} \in V$ and $c \in F$. We have

$$O(\mathbf{u} + \mathbf{v}) = \mathbf{0} = \mathbf{0} + \mathbf{0} = O(\mathbf{u}) + O(\mathbf{v})$$

and

$$O(c\mathbf{v}) = \mathbf{0} = c\mathbf{0} = cO(\mathbf{v}).$$

Thus O is a linear map.

(vi) (This question requires integration). Define $T : P(\mathbb{R}) \to \mathbb{R}$ by

$$T(p(x)) = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Solution: Let $p(x), q(x) \in P(\mathbb{R})$ and $c \in \mathbb{R}$. Then using theorems on integration from calculus, we have

$$\begin{aligned} T(p(x) + q(x)) &= \int_0^1 (p(x) + q(x))dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p(x)) + T(q(x)) \end{aligned}$$

and

$$\begin{aligned} T(cp(x)) &= \int_0^1 cp(x)dx \\ &= c\int_0^1 p(x)dx \\ &= cT(p(x)). \end{aligned}$$

Thus T is linear.

(vii) For each function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given below, show that T is not a linear map.

(a) $T(x, y) = (xy, y)$

(b) $T(x, y) = (e^x, e^y)$

(c) $T(x, y) = (1, 0)$

Solution: This problem is also solved on MS-LAP. Any one counterexample will show that T is not a linear map. We just give one such counterexample here.

(a) We have $T(1, 1) = (1, 1)$, so $2T(1, 1) = (2, 2)$. However, $T(2(1, 1)) = T(2, 2) = (4, 2) \neq 2T(1, 1)$. Thus T is not linear.

(b) Since $T(0, 0) = (1, 1) \neq (0, 0)$, T is not linear (a linear map must send $\mathbf{0}$ to $\mathbf{0}$).

(c) Since $T(0, 0) = (1, 0) \neq (0, 0)$, T is not linear.

(viii) Let V and W be vector spaces over the same field F, and let $T : V \to W$ be a linear map.

(a) Show that $T(-\mathbf{v}) = -T(\mathbf{v})$.

(b) Use part (a) to deduce that $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$.

Solution: This problem is also solved on MS-LAP.

(a) Taking $c = 1$ in $T(c\mathbf{v}) = cT(\mathbf{v})$ yields the result (we have previously seen that $(-1)\mathbf{v} = -\mathbf{v}$).

(b) We have

$$\begin{aligned} T(\mathbf{u} - \mathbf{v}) &= T(\mathbf{u} + (-\mathbf{v})) \\ &= T(\mathbf{u}) + T(-\mathbf{v}) \qquad (T \text{ is linear}) \\ &= T(\mathbf{u}) - T(\mathbf{v}) \qquad (\text{from part (a)}). \end{aligned}$$

(ix) Let V and W be vector spaces over the same field F. Let $S : V \to W$ and $T : V \to W$ be linear maps.

(a) Define the function $S + T : V \to W$ by $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$. Show that $S + T$ is linear.

(b) Let $c \in F$. Define the function $cT : V \to W$ by $(cT)(\mathbf{v}) = cT(\mathbf{v})$. Show that cT is linear.

(c) (This question is a bit long). Let $L(V, W)$ be the set of all linear maps from V to W. Show that $L(V, W)$ is a vector space over F with operations of addition and scalar multiplication as defined in parts (a) and (b).

Solution:

(a) Let $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in F$. Then

$$\begin{aligned} (S + T)(\mathbf{u} + \mathbf{v}) &= S(\mathbf{u} + \mathbf{v}) + T(\mathbf{u} + \mathbf{v}) \qquad (\text{def. of } S + T) \\ &= (S(\mathbf{u}) + S(\mathbf{v})) + (T(\mathbf{u}) + T(\mathbf{v})) \qquad (S \text{ and } T \text{ are linear}) \\ &= (S(\mathbf{u}) + T(\mathbf{u})) + (S(\mathbf{v}) + T(\mathbf{v})) \end{aligned}$$

$$= (S+T)(\mathbf{u}) + (S+T)(\mathbf{v}) \qquad \text{(def. of } S+T\text{)}$$

and

$$\begin{aligned}(S+T)(\alpha\mathbf{v}) &= S(\alpha\mathbf{v}) + T(\alpha\mathbf{v}) && \text{(def. of } S+T\text{)}\\ &= \alpha S(\mathbf{v}) + \alpha T(\mathbf{v}) && (S \text{ and } T \text{ are linear})\\ &= \alpha\left(S(\mathbf{v}) + T(\mathbf{v})\right)\\ &= \alpha(S+T)(\mathbf{v}) && \text{(def. of } S+T\text{).}\end{aligned}$$

Thus $S+T$ is linear.

(b) Let $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in F$. Then

$$\begin{aligned}(cT)(\mathbf{u}+\mathbf{v}) &= cT(\mathbf{u}+\mathbf{v}) && \text{(def. of } cT\text{)}\\ &= c\left(T(\mathbf{u}) + T(\mathbf{v})\right) && (T \text{ is linear})\\ &= cT(\mathbf{u}) + cT(\mathbf{v})\\ &= (cT)(\mathbf{u}) + (cT)(\mathbf{v}) && \text{(def. of } cT\text{)}\end{aligned}$$

and

$$\begin{aligned}(cT)(\alpha\mathbf{v}) &= cT(\alpha\mathbf{v}) && \text{(def. of } cT\text{)}\\ &= c(\alpha T(\mathbf{v})) && (T \text{ is linear})\\ &= \alpha(cT(\mathbf{v}))\\ &= \alpha\left((cT)(\mathbf{v})\right) && \text{(def. of } cT\text{).}\end{aligned}$$

Thus cT is linear.

(c) We must show that all vector space axioms hold for $L(V, W)$. Our vectors here are linear maps from V to W. Note that for $T \in L(V, W)$ and $\mathbf{v} \in V$, $T(\mathbf{v})$ is a vector in W. Let R, S, T be linear maps from V to W, and let $b, c \in F$.

(A1): We show that the maps $(R+S)+T$ and $R+(S+T)$ are equal. For and $\mathbf{v} \in V$, we have

$$\begin{aligned}((R+S)+T)(\mathbf{v}) &= (R+S)(\mathbf{v}) + T(\mathbf{v}) && \text{(def. of addition of maps)}\\ &= (R(\mathbf{v}) + S(\mathbf{v})) + T(\mathbf{v}) && \text{(def. of addition of maps)}\\ &= R(\mathbf{v}) + (S(\mathbf{v}) + T(\mathbf{v})) && \text{(axiom A1 of } W\text{)}\\ &= R(\mathbf{v}) + (S+T)(\mathbf{v}) && \text{(def. of addition of maps)}\\ &= (R+(S+T))(\mathbf{v}) && \text{(def. of addition of maps).}\end{aligned}$$

Thus $(R+S)+T = R+(S+T)$.

(A2): For any $\mathbf{v} \in V$ we have

$$\begin{aligned}(S+T)(\mathbf{v}) &= S(\mathbf{v}) + T(\mathbf{v}) && \text{(def. of addition of maps)}\\ &= T(\mathbf{v}) + S(\mathbf{v}) && \text{(axiom A2 of } W\text{)}\\ &= (T+S)(\mathbf{v}) && \text{(def. of addition of maps).}\end{aligned}$$

Thus $S+T = T+S$.

(A3): Let $O : V \to W$ be the zero map. By suggested exercise (v), O is linear, and hence in $L(V, W)$. For any $\mathbf{v} \in V$ we have

$$\begin{aligned}(T+O)(\mathbf{v}) &= T(\mathbf{v}) + O(\mathbf{v}) \qquad \text{(def. of addition of maps)}\\ &= T(\mathbf{v}) + \mathbf{0} \qquad \text{(def. of zero map)}\\ &= T(\mathbf{v}) \qquad \text{(axiom A3 of } W\text{).}\end{aligned}$$

Thus $T + O = T$.

(A4): For linear map T, we define $-T$ by $(-T)(\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v} \in V$. Note that since by vector space properties we have $-\mathbf{u} = (-1)\mathbf{u}$, it follows that $(-T)(\mathbf{v}) = (-1)(T(\mathbf{v}))$. Thus by part (b), $-T$ is also linear, and hence in $L(V, W)$. Then for any $\mathbf{v} \in V$ we have

$$\begin{aligned}(T+(-T))(\mathbf{v}) & T(\mathbf{v}) + (-T)(\mathbf{v}) \qquad \text{(def. of addition of maps)}\\ &= T(\mathbf{v}) - T(\mathbf{v}) \qquad \text{(def. of } -T\text{)}\\ &= \mathbf{0} \qquad \text{(axiom A4 of } W\text{)}\\ &= O(\mathbf{v}) \qquad \text{(def. of zero map).}\end{aligned}$$

Thus $T + (-T) = O$.

(S1): For any $\mathbf{v} \in V$ we have

$$\begin{aligned}(c(S+T))(\mathbf{v}) &= c\left((S+T)(\mathbf{v})\right) \qquad \text{(def. of scalar mult. of maps)}\\ &= c\left(S(\mathbf{v}) + T(\mathbf{v})\right) \qquad \text{(def. of addition of maps)}\\ &= cS(\mathbf{v}) + cT(\mathbf{v}) \qquad \text{(axiom S1 of } W\text{)}\\ &= (cS)(\mathbf{v}) + (cT)(\mathbf{v}) \qquad \text{(def. of scalar mult of maps)}\\ &= (cS + cT)(\mathbf{v}) \qquad \text{(def. of addition of maps).}\end{aligned}$$

This $c(S + T) = cS + cT$.

(S2): For any $\mathbf{v} \in V$ we have

$$\begin{aligned}((b+c)T)(\mathbf{v}) &= (b+c)(T(\mathbf{v})) \qquad \text{(def. of scalar mult. of maps)}\\ &= bT(\mathbf{v}) + cT(\mathbf{v}) \qquad \text{(axiom S2 of } W\text{)}\\ &= (bT)(\mathbf{v}) + (cT)(\mathbf{v}) \qquad \text{(def. of scalar mult. of maps)}\\ &= (bT + cT)(\mathbf{v}) \qquad \text{(def. of addition of maps).}\end{aligned}$$

Thus $(b + c)T = bT + cT$.

(S3): For any $\mathbf{v} \in V$ we have

$$\begin{aligned}((bc)T)(\mathbf{v}) &= (bc)T(\mathbf{v}) \qquad \text{(def. of scalar mult. of maps)}\\ &= b(cT(\mathbf{v})) \qquad \text{(axiom S3 of } W\text{)}\\ &= b((cT)(\mathbf{v})) \qquad \text{(def. of scalar mult. of maps)}\\ & (b(cT))(\mathbf{v}) \qquad \text{(def. of scalar mult. of maps).}\end{aligned}$$

Thus $(bc)T = b(cT)$.

(S4): Recall that $1 \in F$ is the multiplicative identity of F. For any $\mathbf{v} \in V$ we have

$$\begin{aligned}(1T)(\mathbf{v}) &= 1T(\mathbf{v}) \qquad \text{(def. of scalar mult. of maps)}\\ &= T(\mathbf{v}) \qquad \text{(axiom S4 of } W\text{).}\end{aligned}$$

Thus $1T = T$.

(x) In this question, we show neither of the two conditions for linear maps implies the other. That is, dropping one of the two conditions will allow maps that are not linear.

(a) Consider $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = \sqrt[3]{x^2y}$. Prove that for any $\mathbf{v} \in \mathbb{R}^2$ and any $c \in \mathbb{R}$, we have $T(c\mathbf{v}) = cT(\mathbf{v})$. Then demonstrate that $T((1, 0) + (0, 1)) \neq T(1, 0) + T(0, 1)$ (for example) to show that T is not a linear map.

(b) Consider $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = \overline{z}$ (recall that $\overline{z}$ is the conjugate of z). Prove that for any $z_1, z_2 \in \mathbb{C}$, we have $T(z_1 + z_2) = T(z_1) + T(z_2)$. Then demonstrate that $T(cz) \neq cT(z)$ for $c = i$ and $z = 1$ (for example) to show that T is not a linear map.

Solution:

(a) We have

$$\begin{aligned}T(c(x, y)) &= T(cx, cy)\\ &= \sqrt[3]{(cx)^2(cy)}\\ &= \sqrt[3]{c^3x^2y}\\ &= c\sqrt[3]{x^2y}\\ &= cT(x, y).\end{aligned}$$

However, $T((1, 0) + (0, 1)) = T(1, 1) = 1$ and $T(1, 0) + T(0, 1) = 0 + 0 = 0$ and not equal. Thus the first property of linear maps fails, and hence T is not linear.

(b) Let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ for $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then

$$\begin{aligned}T(z_1 + z_2) &= T((a_1 + a_2) + (b_1 + b_2)i)\\ &= (a_1 + a_2) - (b_1 + b_2)i\\ &= (a_1 - b_1i) + (a_2 - b_2i)\\ &= T(z_1) + T(z_2).\end{aligned}$$

However, $T(i \cdot 1) = T(i) = -i$ and $iT(1) = i(1) = i$ are not equal. Thus the second property of linear maps fails, and hence T is not linear.