MATH 1102 Fall & Winter 2017/2018

Assignment 10

Due Tuesday January 23 in class.

Note: When a basis is given, we adopt the convention that it is an ordered basis, where the order of the basis elements is given by the order in which they are written in the set. For example, $B_1 = \{(1,2),(1,3)\}$ is a basis of $\mathbb{R}^2$ with first element $(1,2)$, while $B_2 = \{(1,3),(1,2)\}$ is a basis with first element $(1,3)$ (even though B_1 and B_2 are equal as sets).

Note: By now, you are experts at row reducing matrices. To save you some time, on this and future assignments (but not on tests) you may use a matrix calculator to row reduce any matrices that consist entirely of numbers, and do not have to show the details of the calculation. If a matrix contains both numbers and parameters or variables, then you should complete the row reduction by hand. There are many row reduction calculators online. For example, in Wolfram Alpha (wolframalpha.com), typing "row reduce {{1,2},{1,3}}" (without the quotes) will give the RREF of the 2×2 matrix with *rows* $(1,2)$ and $(1,3)$.

The following questions are to be turned in to be graded.

1. Consider $\mathbf{v} = (2,3,4) \in \mathbb{R}^3$ (over $\mathbb{R}$). In each part below, we give an ordered basis of $\mathbb{R}^3$ (you do *not* have to prove that they are bases here). In parts (a) to (c), you are asked to find coordinate vectors. If you can find a coordinate vector by inspection, that is fine. Just briefly show that it works.

   (a) (1 mark) If $B_1 = \{(1,1,1),(0,1,1),(0,0,1)\}$, then find $[\mathbf{v}]_{B_1}$.

   (b) (0.5 marks) If $B_2 = \{(0,0,1),(0,1,1),(1,1,1)\}$, then find $[\mathbf{v}]_{B_2}$.

   (c) (1 mark) If $B_3 = \{(1,0,1),(1,1,1),(1,2,3)\}$, then find $[\mathbf{v}]_{B_3}$

   (d) (0.5 marks) If $[\mathbf{u}]_{B_3} = (3,-1,2)$ with B_3 as in part (c), find $\mathbf{u}$.

2. Let V be an finite dimensional vector space over F with ordered basis B.

   (a) (1 mark) Prove $[\mathbf{0}]_B = \mathbf{0}$ (note that the zero vector on the left side is the zero vector in V, while the zero vector on the right side is the zero vector in F^n).

   (b) (3 marks) Prove that if $[\mathbf{v}_1]_B, ..., [\mathbf{v}_k]_B$ are linearly independent in F^n, then $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent in V.

3. (3 marks) In this question, consider $P_2(\mathbb{R})$ with ordered basis $B = \{x^2, x, 1\}$.

   (a) Write down the coordinate vectors (with respect to B) for the polynomials x^2+2x+1, x^2+x+2, and x^2+4x-1. Form the matrix with columns equal to these coordinate vectors and put it in reduced row echelon form. Are the polynomials linearly dependent or independent? If they are linearly dependent, write down an explicit dependence relation.

(b) Write down the coordinate vectors (with respect to B) for the polynomials $x^2 + 2x + 1$, $x^2 + x + 2$, and $x^2 + 4x + 1$ (only the constant term in the last polynomial has been changed from part (a)). Form the matrix with columns equal to these coordinate vectors and put it in reduced row echelon form. Are the polynomials linearly dependent or independent? If they are linearly dependent, write down an explicit dependence relation.

4. (1 mark) Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x, y) = (2x - 3y, -5x + y, 7x - y)$. Find a matrix A such that

$$T \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then use a theorem from class about matrix mappings to deduce that T is a linear map.

5. Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(ax^2 + bx + c) = a + b + c$.

   (a) (0.5 marks) Find $T(4x^2 - 7x + 9)$.

   (b) (2 marks) Show that T is a linear map.

   (c) (0.5 marks) Describe the set $V = \{\mathbf{v} \in P_2(\mathbb{R}) : T(\mathbf{v}) = \mathbf{0}\}$. This set should be familiar to you. Is V a subspace of $P_2(\mathbb{R})$? You may refer to any previously seen work, such as material on the midyear test, to answer this question.

6. In each part below, we give a function $T : P(\mathbb{R}) \to P(\mathbb{R})$.

   (a) (1.5 marks) Define T by $T(p(x)) = xp(x)$. Show that T is a linear map.

   (b) (1 mark) Define T by $T(p(x)) = (p(x))^2$. Show that T is not a linear map.

   (c) (1.5 marks) Define T by $T(p(x)) = p(x^2)$. Show that T is a linear map. Hint: note that if $p(x) = \sum_{i=0}^{n} a_i x^i$, then $T(p(x)) = \sum_{i=0}^{n} a_i x^{2i}$.

7. Recall that we proved that a linear map $T : V \to W$ is completely determined by where it sends the elements of a basis of V.

   (a) (1 mark) Suppose $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map that satisfies $T_1(1, 0) = (2, -3)$ and $T_1(0, 1) = (-1, 2)$. Find an explicit formula for $T_1(x, y)$.

   (b) (1 mark) Suppose $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map that satisfies $T_2(1, 0) = (1, 1)$ and $T_2(0, 1) = (1, 1)$. Find an explicit formula for $T_2(x, y)$. Note that T_2 sends both basis vectors to $(1, 1)$. Does this mean that T_2 sends all elements of $\mathbb{R}^2$ to $(1, 1)$?

   (c) (2 marks) Suppose $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map that satisfies $T_3(1, 1) = (2, -3)$ and $T_3(1, -1) = (-1, 5)$. Find an explicit formula for $T_3(x, y)$. Hint: one way to proceed is to find c_1 and c_2 with $(x, y) = c_1(1, 1) + c_2(1, -1)$ (c_1 and c_2 will depend on x and y), and then use the fact that T is linear.

8. Let V and W be vector spaces over the same field F. This question shows that the two conditions for a linear map may be replaced with a single condition.

(a) (2 marks) Suppose $T : V \to W$ is a function such that for any $c \in F$ and any $\mathbf{u}, \mathbf{v} \in V$, we have

$$T(c\mathbf{u} + \mathbf{v}) = cT(\mathbf{u}) + T(\mathbf{v}).$$

Prove that T is a linear map.

(b) (1 mark) Suppose $T : V \to W$ is a linear map. Prove that for any $c \in F$ and any $\mathbf{u}, \mathbf{v} \in V$, we have

$$T(c\mathbf{u} + \mathbf{v}) = cT(\mathbf{u}) + T(\mathbf{v}).$$

The following are *suggested exercises* and are *not* be to turned in.

(i) Let F be a field, and consider the vector space F^n over F. Let $\mathbf{e}_i \in F^n$ be the vector with ith entry equal to 1 and all other entries equal to 0. Then $B = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is the standard basis of F^n (over F). Let $\mathbf{v} \in F^n$. Prove that $[\mathbf{v}]_B = \mathbf{v}$.

(ii) Consider the vector space $M_{22}(\mathbb{R})$ over $\mathbb{R}$. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$.

(a) Find $[A]_{B_1}$, where B_1 is the ordered basis

$$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

(b) Find $[A]_{B_2}$, where B_2 is the ordered basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(iii) Consider the vector space $V = \operatorname{span}\{e^x, e^{-x}\}$ over $\mathbb{R}$ (this is a subspace of the vector space consisting of all functions mapping from $\mathbb{R}$ to $\mathbb{R}$). The hyperbolic sine and cosine functions are defined as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

One may show that e^x and e^{-x} are linearly independent (you do not have to do this here), and so $B = \{e^x, e^{-x}\}$ is an ordered basis for V. Find the coordinate vectors for $\sinh(x)$ and $\cosh(x)$ with respect to B.

(iv) Consider the $V = \operatorname{span}\{x^2, 2^x, \sin x, \cos x\}$. This is a subspace (over $\mathbb{R}$) of the vector space of functions mapping from $\mathbb{R}$ to $\mathbb{R}$. The set $B = \{x^2, 2^x, \sin x, \cos x\}$ is an ordered basis of V (you do not have to show this here).

(a) Find the coordinate vector with respect to the basis B for each of the following elements of V:

$$\begin{aligned}
\mathbf{v}_1 &= x^2 + \sin x - \cos x \\
\mathbf{v}_2 &= 2^x + 2\sin x + 2\cos x \\
\mathbf{v}_3 &= 2x^2 - 2^x - 4\cos x \\
\mathbf{v}_4 &= x^2 + 2^x + \sin x + \cos x.
\end{aligned}$$

(b) Using the coordinate vectors from part (a), determine if the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent or linearly independent. If they are linearly dependent, write down an explicit dependence relation.

(c) Using the coordinate vectors from part (a), determine if the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_4$ are linearly dependent or linearly independent. If they are linearly dependent, write down an explicit dependence relation.

(v) Let V and W be vector spaces over the same field F. Consider the zero map $O : V \to W$ defined by $O(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. Show that O is a linear map.

(vi) (This question requires integration). Define $T : P(\mathbb{R}) \to \mathbb{R}$ by

$$T(p(x)) = \int_0^1 p(x)dx.$$

Show that T is a linear map.

(vii) For each function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given below, show that T is not a linear map.

(a) $T(x, y) = (xy, y)$

(b) $T(x, y) = (e^x, e^y)$

(c) $T(x, y) = (1, 0)$

(viii) Let V and W be vector spaces over the same field F, and let $T : V \to W$ be a linear map.

(a) Show that $T(-\mathbf{v}) = -T(\mathbf{v})$.

(b) Use part (a) to deduce that $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$.

(ix) Let V and W be vector spaces over the same field F. Let $S : V \to W$ and $T : V \to W$ be linear maps.

(a) Define the function $S + T : V \to W$ by $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$. Show that $S + T$ is linear.

(b) Let $c \in F$. Define the function $cT : V \to W$ by $(cT)(\mathbf{v}) = cT(\mathbf{v})$. Show that cT is linear.

(c) (This question is a bit long). Let $L(V, W)$ be the set of all linear maps from V to W. Show that $L(V, W)$ is a vector space over F with operations of addition and scalar multiplication as defined in parts (a) and (b).

(x) In this question, we show neither of the two conditions for linear maps implies the other. That is, dropping one of the two conditions will allow maps that are not linear.

(a) Consider $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = \sqrt[3]{x^2 y}$. Prove that for any $\mathbf{v} \in \mathbb{R}^2$ and any $c \in \mathbb{R}$, we have $T(c\mathbf{v}) = cT(\mathbf{v})$. Then demonstrate that $T((1, 0) + (0, 1)) \neq T(1, 0) + T(0, 1)$ (for example) to show that T is not a linear map.

(b) Consider $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = \overline{z}$ (recall that $\overline{z}$ is the conjugate of z). Prove that for any $z_1, z_2 \in \mathbb{C}$, we have $T(z_1 + z_2) = T(z_1) + T(z_2)$. Then demonstrate that $T(cz) \neq cT(z)$ for $c = i$ and $z = 1$ (for example) to show that T is not a linear map.

For your reference:

Let V be a finite dimensional vector space over a field F with $\dim(V) = n$. Let $B = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be an ordered basis for V (an ordered basis means that we have chosen some element in the basis to be $\mathbf{v}_1$, another to be $\mathbf{v}_2$, and so on). Let $\mathbf{v} \in V$. Then there exist unique $c_i \in F$ such that $\mathbf{v} = c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n$. The *coordinate vector for* $\mathbf{v}$ *with respect to* B is $[\mathbf{v}]_B = (c_1, ..., c_n)$. It is an element of F^n. We saw the following results about coordinate vectors:

- For all $\mathbf{u}, \mathbf{v} \in V$ and all $c \in F$, we have $[\mathbf{u}+\mathbf{v}]_B = [\mathbf{u}]_B + [\mathbf{v}]_B$ and $[c\mathbf{v}]_B = c[\mathbf{v}]_B$.
- $[\mathbf{0}]_B = \mathbf{0}$ (the proof of this was left as an exercise and is on this assignment). Note that the zero vector on the left hand side is in V, while the zero vector on the right hand side is in F^n.
- $[\mathbf{u}]_B = [\mathbf{v}]_B$ if and only if $\mathbf{u} = \mathbf{v}$ (the proof of this was left as an exercise and is one of this week's tutorial problems).
- Vectors $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent in V if and only if $[\mathbf{v}_1]_B, ..., [\mathbf{v}_k]_B$ are linearly independent in F^n (part of the proof of this theorem was left as an exercise, and is on this assignment).
- Vectors $\mathbf{v}_1, ..., \mathbf{v}_k$ span V if and only if $[\mathbf{v}_1]_B, ..., [\mathbf{v}_k]_B$ span F^n.

Let V and W be vector spaces over the same field F. A function $T : V \to W$ is a *linear map* if

1. For all $\mathbf{u}, \mathbf{v} \in V$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
2. For all $c \in F$ and all $\mathbf{v} \in V$, $T(c\mathbf{v}) = cT(\mathbf{v})$.

Let $A \in M_{mn}(F)$. We proved that the function $T : F^n \to F^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$ is a linear map.

Let V and W be vector spaces over the same field F with V finite dimensional. Let $\mathbf{v}_1, ..., \mathbf{v}_n$ be a basis of V, and let $\mathbf{w}_1, ..., \mathbf{w}_n$ be *any* vectors in W. We proved that there is a unique linear map $T : V \to W$ satisfying $T(\mathbf{v}_i) = \mathbf{w}_i$ for each $i = 1, ..., n$. Note that there are no restrictions on the $\mathbf{w}_i$; they may be linearly dependent, or may even be the same vector. This theorem shows that a linear map is completely determined by where it sends a basis.