This test paper has 6 questions (3 multiple choice and 3 long answer). It can not be taken from the examination room. Total of 30 marks. No calculators are allowed. Duration: 50 Minutes.

NAME: **STUDENT NO:**

Multiple Choice Questions. Circle the correct answer. No partial marks.

[2] **1.** Suppose a 9×6 matrix A has 4 pivot columns. What is $\dim(\text{Nul}A)$?

(a) 3 (b) 6 (c) 4 (d) 5 (e) 2

[2] **2.** You are given that $\mathcal{B} = \{1 - x^2, 1 + x, 1 - x - x^2\}$ is a basis for $\mathbb{P}_2$.

If $p(x) = \alpha + \beta x + \gamma x^2$ and the $\mathcal{B}$-coordinate vector of $p(x)$ is $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$,

what is the value of β?

(a) 2 (b) 3 (c) 6 (d) 5 (e) -11

$2(1-x^2)+6(1+x)+3(1-x-x^2)$
$= 2+6+3+(6-3)x+(-2-3)x^2 = 11+3x-5x^2$

[2] **3.** Let $T : \mathbb{R}^2 \longrightarrow \mathbb{P}_2$ be a linear transformation defined by

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1 + 2x + 3x^2, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2 + 3x + x^2.$$

If $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \alpha + \beta x + \gamma x^2$, what is the value of γ ?

(a) 11 (b) 10 (c) 9 (d) 8 (e) 5

$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = 3\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
$= 3(1+2x+3x^2) + 2(2+3x+x^2)$
$= (3+4)+(6+6)x+(9+2)x^2$
$= 7+12x+11x^2$

Long answer questions. Show all your work.

[7] [~~6~~] **4.** Define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(a+bx+cx^2) = \begin{bmatrix} a+2b+c \\ a+b+5c \\ +b-4c \end{bmatrix}$.

(a) Find a basis for $\ker(T)$.

$$T(a+bx+cx^2)=0 \Rightarrow \begin{bmatrix} a+2b+c \\ a+b+5c \\ b-4c \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 5 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -9c \\ 4c \\ c \end{bmatrix} = c\begin{bmatrix} -9 \\ 4 \\ 1 \end{bmatrix}$$ 3 (1 mark for each correct coordinate)

$\ker(T) = \text{span}\{-9+4x+x^2\}$

$\{p(x) = -9+4x+x^2\}$ is a basis for $\ker(T)$. 2

Find a basis for Range(T)

(b) ~~What is the dimension of Range(T)? Give your reason~~.

$$\text{Range}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}\right\}$$

$$= \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{u}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{v}\right\}$$

A basis for Range$(T) = \{u, v\}$ 2

dim Range$(T) = 2$.

[6] [8] **5.** You are given that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Find the coordinate vector of $u = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ with respect to basis $\mathcal{B}$.

$$\left[\begin{array}{ccc|c} 1 & -5 & 3 & -2 \\ -1 & 6 & -2 & 5 \\ 1 & -4 & 5 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 3 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 3 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad 3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 0 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$[u]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}. \quad 3$$

(1 mark for each correct entry)

<u>Verify:</u> $-3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 6 \\ -4 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -3-5+6 \\ 3+6-4 \\ -3-4+10 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = u$ ✓

[10] **6.** You are given that $A = \begin{bmatrix} 1 & 4 & -1 & 3 & 0 \\ 1 & 4 & 0 & 0 & 4 \\ 0 & 0 & 1 & -4 & 7 \\ 2 & 8 & -1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$

(a) Find a basis for Col(A). What is ~~the~~ dim(Col(A))?

A basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 3 \end{bmatrix} \right\}$ 1

$\dim \mathrm{Col}(A) = 3$ 1

(b) Express ~~Show that~~ the vector $u = \begin{bmatrix} -4 \\ 3 \\ 9 \\ -1 \end{bmatrix}$ ~~is in ColA. Show your work.~~ as a linear combination of the basis elements of Col A.

1

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & -4 & 9 \\ 2 & -1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & -3 & 7 \\ 0 & 1 & -4 & 9 \\ 0 & 1 & -3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 1$$

$c_3 = -2, \quad c_2 - 3c_3 = 7 \Rightarrow c_2 = 7 + 3c_3 = 7 - 6 = 1$

$c_1 = c_2 - 3c_3 - 4 = 1 + 6 - 4 = 3$ (1 mark for each correct coefficient)

$$u = 3\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 0 \\ -4 \\ 3 \end{bmatrix} \quad 3$$

$\Rightarrow$ u is a linear combination of the basis elements of Col$A$ $\Rightarrow u \in$ Col$A$.

(c) Find a basis for Nul$A$. What is ~~the~~ dim(Nul(A))?

$$Ax = 0 \Rightarrow Bx = 0 \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 4x_5 \\ x_2 \\ 5x_5 \\ 3x_5 \\ x_5 \end{bmatrix} \quad 1$$

A basis for Nul$A$ is

$\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 5 \\ 3 \\ 1 \end{bmatrix} \right\}$ 2 $\qquad \dim \mathrm{Nul}A = 2.$ 1