CONCORDIA UNIVERSITY **Department of Mathematics and Statistics**

Course	Number		Section(s)
Mathematics	**MATH 264 / MAST 218**		**AA**
Examination	Date	Time	Pages
Final	**April 2016**	**3 hours**	**9**
Instructors			Course Examiner
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Special Instructions: Approved calculators permitted. Lined paper booklets.

READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!! JUSTIFY ALL STEPS !!! GOOD LUCK !!!

MARKS: marks for each problem are shown in front of the problems. Maximum total is 100.

↓MARKS

10 **1)** Find the area of the region that lies inside the curve $r = 3\sin\theta$ and outside of the curve $r = 2 - \sin\theta$, where (r, θ) are polar coordinates. First, sketch the curves.

Solution:

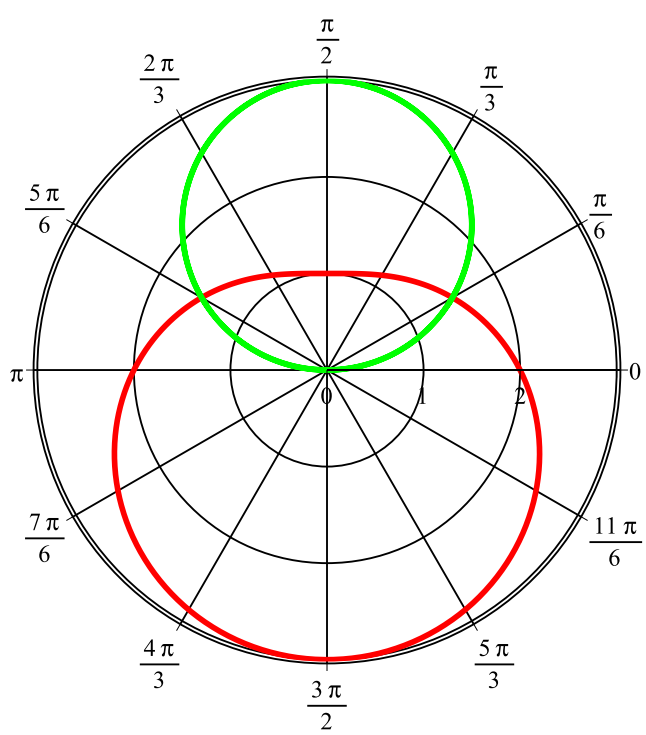


FIGURE 1. Problem 1)

Where the curves intersect? $3\sin\theta = 2 - \sin\theta$ gives $\sin\theta = 1/2$ or $\theta = \pi/6, \pi - \pi/6$. Thus, the required area is

$$A = 2 \cdot \frac{1}{2} \cdot \int_{\pi/6}^{\pi/2} [(3\sin\theta)^2 - (2-\sin\theta)^2] d\theta = \int_{\pi/6}^{\pi/2} [8\sin^2\theta + 4\sin\theta - 4] d\theta$$

We have

$$8\int_{\pi/6}^{\pi/2} \sin^2\theta d\theta = 8\int_{\pi/6}^{\pi/2} [1/2 - 1/2\cos(2\theta)] d\theta$$

$$= 4(\pi/3) - [4 \cdot \frac{1}{2}\sin(2\theta)]_{\pi/6}^{\pi/2} = 4(\pi/3) + 2\sin(\pi/3) = 4\pi/3 + 2\sqrt{3}/2.$$

$$\int_{\pi/6}^{\pi/2} [4\sin\theta - 4] d\theta = -4[\cos(\theta)]_{\pi/6}^{\pi/2} - 4\pi/3 = 4\sqrt{3}/2 - 4\pi/3.$$

Thus,

$$A = 3\sqrt{3}.$$

2)

4 (a) At what point of the curve $x = 2t^3$, $y = 1 + 4t - t^2$ does the tangent line have slope 1?

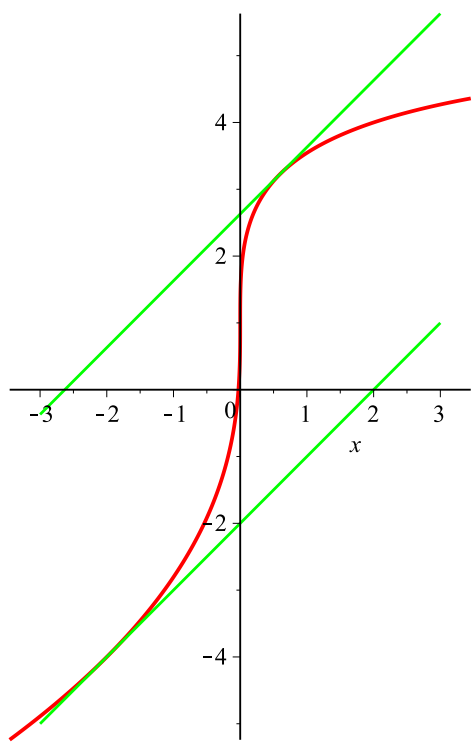


FIGURE 2. Problem 2 a)

Solution: We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2t+4}{6t^2}.$$

$$\frac{-2t+4}{6t^2} = 1 \implies 6t^2 + 2t - 4 = 0 \implies t = -1, t = 2/3.$$

These correspond to points $(-2, -4)$ and $(16/27, 29/9)$.

4 (b) Expand $f(x) = x^2 e^{-x^2}$ in powers of x. Use this expansion to approximate the value of

$$\int_0^{0.5} x^2 e^{-x^2} dx,$$

with error $< 10^{-4}$. Justify your results.

Solution: We have

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots$$

Thus,

$$x^2 e^{-x^2} = x^2(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots) = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \frac{x^{10}}{4!} - \frac{x^{12}}{5!} + \dots$$

and

$$\int x^2 e^{-x^2} dx = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} - \frac{x^{13}}{13 \cdot 5!} + \dots$$

and

$$\int_0^{0.5} x^2 e^{-x^2} dx = \frac{0.5^3}{3} - \frac{0.5^5}{5} + \frac{0.5^7}{7 \cdot 2!} - \frac{0.5^9}{9 \cdot 3!} + \frac{0.5^{11}}{11 \cdot 4!} - \frac{0.5^{13}}{13 \cdot 5!} + \dots$$

Since this is an alternating series the error after dropping the tail is smaller than the first dropped term. We need this term to be $< 10^{-4}$. We have

$$a_1 = \frac{0.5^3}{3} \sim 0.04166666667,$$

$$a_2 = -\frac{0.5^5}{5} \sim -0.006250000000,$$

$$a_3 = \frac{0.5^7}{7 \cdot 2!} \sim 0.0005580357143,$$

$$a_4 = -\frac{0.5^9}{9 \cdot 3!} \sim -0.00003616898148.$$

Since $|a_4| < 0.0001$ the required approximation is

$$a_1 + a_2 + a_3 \sim 0.03597470238.$$

3) Find the limit or show that it does not exist:

5 (a)

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}.$$

Solution: Introducing $t = x^2 + y^2$ we have

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} = \lim_{t\to 0^+} \frac{e^{-t} - 1}{t} = \text{l'Hospital} = \lim_{t\to 0^+} \frac{-e^{-t}}{1} = 1.$$

5 (b)

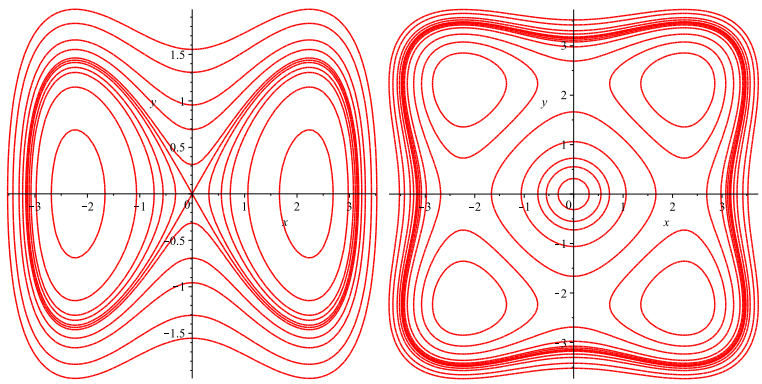
$$\lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2 + y^2}.$$

Solution: We have

$$0 \leq \left| \frac{yx^2}{x^2+y^2} \right| \leq |y|,$$

where we used the fact that $x^2 \leq x^2 + y^2$. Since $|y| \to 0$, our limit also is equal to 0.

5 **4)** One of the contour plots in Figure 3 corresponds to the function $f(x,y) = x^2 - y^2 - \frac{1}{10}x^4 - \frac{1}{10}y^4$. Which one? Justify Your answer.



A B

FIGURE 3. Plots for Problem 4

Solution: The answer is A. The plot A shows a saddle point at $(0,0)$, while plot B shows there a maximum or minimum. We will show that function f has a saddle point at $(0,0)$. We have:

$$\partial f/\partial x = 2x - (4/10)x^3 \quad , \quad \partial f/\partial y = -2y - (4/10)y^3,$$

so $\nabla f(0,0) = (0,0)$. Then,

$$\partial^2 f/\partial x^2 = 2 - (12/10)x^2 \quad , \quad \partial^2 f/\partial y^2 = -2 - (12/10)y^2 \quad , \quad \partial^2 f/\partial y \partial x = \partial^2 f/\partial x \partial y = 0,$$

and the matrix of second derivatives at $(0,0)$ is

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

By the second derivative test, f has a saddle point at $(0,0)$.

5) Let $P = (1,1,1)$, and let L be the line given parametrically

$$L: \quad (0,1,0) + t\langle 1,2,2\rangle, \quad -\infty < t < +\infty.$$

4 (a) Find the distance of point P from line L;

Solution: Point $A = (0,1,0) \in L$. We will calculate the projection $\vec{u}$ of the vector $\vec{v} = \overrightarrow{AP} = \langle 1,0,1\rangle$ onto directional vector $\vec{w} = \langle 1,2,2\rangle$ of line L. We have

$$\vec{u} = (\vec{v}\cdot\vec{w})\vec{w}/|\vec{w}|^2 = (3/9)\langle 1,2,2\rangle.$$

The distance of P from line L is

$$d = \sqrt{|\vec{v}|^2 - |\vec{u}|^2} = \sqrt{2-1} = 1.$$

4 (b) Find the equation of the plane Q passing through point P and the line L;

Solution: Both vectors $\vec{v}$ and $\vec{w}$ are parallel to Q. Thus, the normal vector $\vec{n}$ is

$$\vec{n} = \vec{v} \times \vec{w} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \langle -2, -1, 2 \rangle.$$

Thus, the equation of the plain Q is

$$\langle -2, -1, 2 \rangle \cdot \langle x - 0, y - 1, z - 0 \rangle = 0,$$

or

$$-2x - y + 2z + 1 = 0.$$

4 (c) Find the symmetric equations of the line L_2 passing through point P and perpendicular to the plane Q;

Solution: The normal vector of Q is the directional vector of L_2 so the parametric representation of L_2 is

$$(1, 1, 1) + t\langle -2, -1, 2 \rangle.$$

Thus, the symmetric equations of L_2 are

$$\frac{x-1}{-2} = \frac{y-1}{-1} = \frac{z-1}{2}.$$

4 (d) Find the distance between the lines L and L_2.

Solution: Line L is in Q and L_2 is perpendicular to Q. The distance between these lines is equal to the distance of the point of intersection of L_2 and Q from L, i.e., the distance from P to L which was calculated in part (a) and is equal to 1.

6) Consider the curve $\gamma = (\sqrt{3} \cdot t, \sin(t), \cos(-t))$, $0 \le t \le 1$.

4 (a) Find the unit tangent $\vec{T}$ and unit normal $\vec{N}$ vectors of γ;

Solution: Since $\cos(-t) = \cos(t)$ for all t we can write $\gamma = (\sqrt{3} \cdot t, \sin(t), \cos(t))$. We have $\gamma' = (\sqrt{3}, \cos(t), -\sin(t))$, so $|\gamma'| = \sqrt{3+1} = 2$. Thus,

$$\vec{T} = \langle \sqrt{3}/2, \cos(t)/2, -\sin(t)/2 \rangle.$$

We have

$$\vec{T}' = \langle 0, -\sin(t)/2, -\cos(t)/2 \rangle,$$

and $|\vec{T}'| = \sqrt{\sin^2(t)/4 + \cos^2(t)/4} = 1/2$. Thus,

$$\vec{N} = \langle 0, -\sin(t), -\cos(t) \rangle.$$

5 (b) Find the length of the curve γ.

Solution: We have

$$\text{length } \gamma = \int_0^1 |\gamma'(t)| dt = \int_0^1 2dt = 2.$$

7) Consider the function $f(x, y) = 2x^2 + 4xy + y^3 + x + y$:

3 (a) Find the direction of fastest growth of f at the point $P = (0, 1)$. Find the change of rate of f in this direction;

Solution: We have

$$\nabla f = \langle 4x + 4y + 1, 3y^2 + 4x + 1 \rangle.$$

Thus, $\nabla f(0,1) = \langle 5,4\rangle$ is the direction of the fastest growth of f at $P = (0,1)$. The directional derivative in this direction has to be calculated using the unit vector parallel to $\langle 5,4\rangle$, i.e., vector $\vec{v} = \frac{1}{\sqrt{41}}\langle 5,4\rangle$. We have

$$\frac{\partial f}{\partial \vec{v}} = \nabla f(0,1)\cdot \vec{v} = \langle 5,4\rangle \cdot \frac{1}{\sqrt{41}}\langle 5,4\rangle = \sqrt{41}.$$

3 (b) Find the equation of the plane tangent to the graph of f at the point P;

Solution: We have $f(0,1) = 2$ so the equation of the tangent plane is

$$z = \nabla f(0,1)\cdot\langle x-0, y-1\rangle + 2,$$

or

$$z = 5x + 4y - 2.$$

3 (c) Find the equation of the line tangent to the level curve of f passing through the point P;

Solution: Since the gradient is perpendicular to level curves the required equation is

$$\nabla f(0,1)\cdot\langle x-0, y-1\rangle = 0,$$

or

$$5x + 4y - 4 = 0.$$

10 (d) Find the local minima and local maxima of the function f;

Solution: We find critical points solving $\nabla f = 0$. We have:

$$\begin{cases} 4x + 4y + 1 &= 0, \\ 3y^2 + 4x + 1 &= 0. \end{cases}$$

Subtracting the first from the second we obtain

$$3y^2 - 4y = 0,$$

which gives $y = 0$ or $y = 4/3$. If $y = 0$, then $x = -1/4$. If $y = 4/3$, then $x = -19/12$. The critical points are $(-1/4, 0)$ and $(-19/12, 4/3)$. The matrix of second derivatives:

$$\partial^2 f/\partial x^2 = 4 \quad , \quad \partial^2 f/\partial y^2 = 6y \quad , \quad \partial^2 f/\partial y\partial x = \partial^2 f/\partial x\partial y = 4,$$

so

$$D = \begin{bmatrix} 4 & 4 \\ 4 & 6y \end{bmatrix}.$$

At $(-1/4, 0)$ we have

$$D = \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix},$$

with $\det D = -16$ so it is a saddle point.

At $(-19/12, 4/3)$ we have

$$D = \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix},$$

with $\det D = 16$. By second derivative test it is a local minimum of f.

3 (e) If $x = st$ and $y = s + t$, find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ at point P.

Solution: We have $\dfrac{\partial f}{\partial x}(0,1) = 5$ and $\dfrac{\partial f}{\partial y}(0,1) = 4$. If

$$\begin{cases} 0 &= st, \\ 1 &= s + t, \end{cases}$$

then either $s = 0, t = 1$ or $s = 1, t = 0$.

We have $\dfrac{\partial x}{\partial s} = t$ and $\dfrac{\partial x}{\partial t} = s$.

We have $\dfrac{\partial y}{\partial s} = 1$ and $\dfrac{\partial y}{\partial t} = 1$.

Then,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

For $(s,t) = (0,1)$ we obtain $\dfrac{\partial f}{\partial s} = 5 \cdot 1 + 4 \cdot 1 = 9$

For $(s,t) = (1,0)$ we obtain $\dfrac{\partial f}{\partial s} = 5 \cdot 0 + 4 \cdot 1 = 4$.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

For $(s,t) = (0,1)$ we obtain $\dfrac{\partial f}{\partial t} = 5 \cdot 0 + 4 \cdot 1 = 4$

For $(s,t) = (1,0)$ we obtain $\dfrac{\partial f}{\partial t} = 5 \cdot 1 + 4 \cdot 1 = 9$.

10 **8)** Find the absolute maximum and minimum of $f(x,y) = (x-1)^2 y + (y-1)^2$ on the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 4$.

Solution: First we will find critical points of f. $\nabla f = \langle 2(x-1)y, (x-1)^2 + 2(y-1)\rangle$. Solving

$$\begin{cases} 2(x-1)y &= 0, \\ (x-1)^2 + 2(y-1) &= 0, \end{cases}$$

we obtain from the first equation $x = 1$ or $y = 0$. If $x = 1$, then $y = 1$. If $y = 0$, then $x = 1 \pm \sqrt{2}$. We have two critical points: $C_1 = (1,1)$ and $C_2 = (1+\sqrt{2}, 0)$. The third solution point $(1-\sqrt{2}, 0)$ is outside the triangle.

Now, the boundary:

(i): $x = 0$, $0 \le y \le 4$: $\tilde{f}(y) = y + (y-1)^2 = y^2 - y + 1$. We have $\tilde{f}'(y) = 2y - 1$ with critical point $y = 1/2$ so we obtain additional point of interest $P_1 = (0, 1/2)$.

(ii): $y = 0$, $0 \le x \le 4$: $\tilde{f}(x) = 1$ for all x.

(iii): $y = 4 - x$, $0 \le x \le 4$: $\tilde{f}(x) = (x-1)^2(4-x) + (3-x)^2 = -x^3 + 7x^2 - 15x + 13$. We have $\tilde{f}'(y) = -3x^2 + 14x - 15$ with critical points $x = 5/3$ and $x = 3$ so we obtain two additional points of interest $P_2 = (5/3, 7/3)$ and $P_3 = (3,1)$.

We will also consider the vertices $V_1 = (0,0)$, $V_2 = (0,4)$ and $V_3 = (4,0)$.

Now, we compare the values of f:

$f(C1) = 0$, $f(C_2) = 1$, $f(P_1) = 3/4$, $f(\text{side}\{y = 0\}) = 1$, $f(P_2) = 76/27$, $f(P_3) = 4$, $f(V_1) = f(V_3) = 1$, $f(V_2) = 13$.

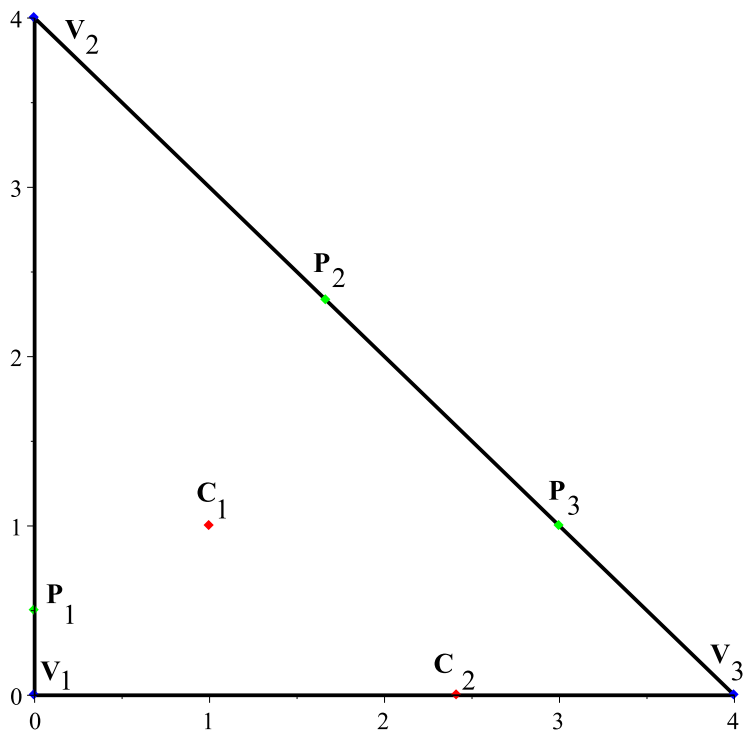


FIGURE 4. Problem 8

We can see that the minimum is 0 at C_1 and the maximum is 13 at V_2.

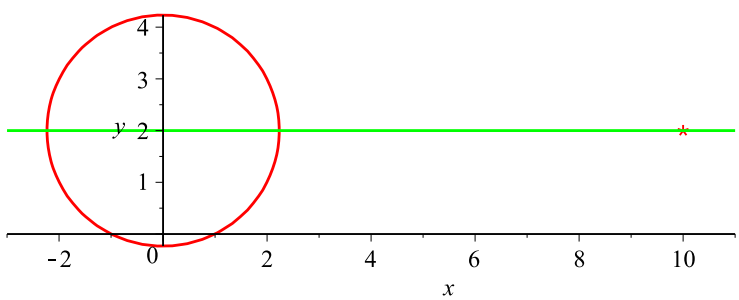


FIGURE 5. Problem 9

10 **9)** Find the distance of the point $P = (10, 2)$ from the curve $x^2 + y^2 - 4y = 1$.

Solution: There are a number of methods to solve this problem. Probably the simplest is to note that our curve is the circle $x^2 + (y-2)^2 = 5$ and make a picture:

The distance is $d = 10 - \sqrt{5}$.

One can also consider the function $d^2(x, y) = (x-10)^2 + (y-2)^2$ (square of the distance from $(10, 2)$) and minimize it on the circle $x^2 + (y-2)^2 = 5$. We can find $(y-2)^2 = 5 - x^2$ and substitute to the function. Then,

$$d^2(x, y) = (x-10)^2 + 5 - x^2 = -20x + 105,$$

with minimum for maximal possible x, i.e., $x = \sqrt{5}$ with $f(\sqrt{5}) = 105 - 20\sqrt{5} = (10 - \sqrt{5})^2$.

One can also use Lagrange method for minimizing d^2.

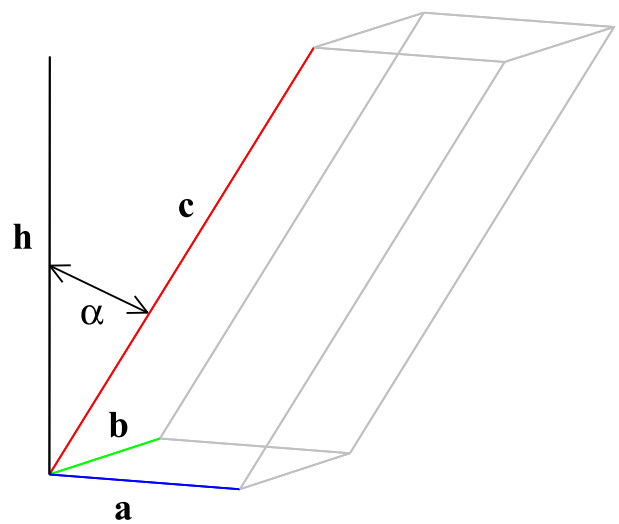


FIGURE 6. Bonus Problem

10 **Bonus:** Prove that $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$ is equal to the volume of the parallelepiped defined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution: We have $|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cdot |\cos\alpha|$ (see Figure 6). $|\vec{a} \times \vec{b}|$ is equal to the area of the parallelogram in the base and $|\vec{c}| \cdot |\cos\alpha|$ is equal to the hight h, so their product gives the volume of the parallelepiped.