Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

CALCULUS I

Instructor: ELIZABETH MALTAIS

MAT1320C – Test 1 – Wednesday, October 10, 2018

- ◁ Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:
- ◁ This is a 75-minute **closed-book** test. No notes. No calculators.
- ◁ **Put away everything** except for a few pens or pencils, an eraser, and your student id. card.
- ◁ The exam consists of 8 questions on 7 pages.
- ◁ maximum points possible = 20 points.
- ◁ Read all questions carefully and be sure to follow the instructions for the individual problems.
- ◁ All questions are long-answer. To receive full marks, your solution must be correct, complete, and show all relevant details.
- ◁ You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.
- ◁ For additional work space, you may use the backs of pages.
  Do not use any of your own scrap paper.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read and understood, and will comply with the above instructions.**

Do not write in this table.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total
Maximum points	2 pts	1 pt	4 pts	4 pts	2 pts	2 pts	2 pts	3 pts	20 points
Marks obtained	2	1	3.5	2	0.5	2	2	3	16

2

Q1. [**2 points**] Using only **the definition**, find the derivative of $f(x) = \sqrt{x+4}$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad f(x+h) = \sqrt{x+h+4}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4} + \sqrt{x+4}}{\sqrt{x+h+4} + \sqrt{x+4}}$$

$$= \lim_{h \to 0} \frac{(x+h+4) - (x+4)}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \frac{1}{\sqrt{x+0+4} + \sqrt{x+4}}$$

$$= \frac{1}{2\sqrt{x+4}}$$

$$f'(x) = \frac{1}{2}(x+4)^{-1/2}$$

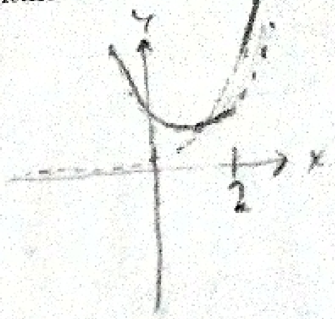
$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

You may double-check your answer using the rules of differentiation:

Q2. [1 point] Give the value(s) of k such that the following function is continuous at all real numbers:

$$g(x) = \begin{cases} (x-1)^2 + k & \text{if } x < 2 \\ 3^x + kx & \text{if } x \geq 2 \end{cases}$$

1



$(x-1)^2 + k = 3^x + kx$

sub in $x = 2$

$(2-1)^2 + k = 3^2 + 2k$

$1 + k = 9 + 2k$

$\boxed{-8 = k}$

$\therefore$ k must equal -8 so that the functions are continuous.

3.5 **Q3.** [4 points] Find each of the following derivatives. You do not need to simplify your answers.

(a) $\dfrac{d}{dt}\left[\dfrac{te^t + 2\pi^3}{t^3 + 2}\right]$

$$\frac{d}{dx}\left[\frac{N(x)}{D(x)}\right] = \frac{N'(x)D(x) - N(x)D'(x)}{(D(x))^2}$$

$$\frac{d}{dt} = \frac{(te^t)(t^3+2) - (te^t + 2\pi^3)(3t^2)}{(t^3+2)^2}$$

use product rule

-0.5

$$\frac{(e^t + te^t)(t^3+2) - (te^t + 2\pi^3)(3t^2)}{(t^3+2)^2}$$

(b) $\dfrac{d}{dx}\left[(\cos(e^x))^5\right]$

$$\frac{d}{dx}\left[(f(x))^n\right] = n(f(x))^{n-1} f'(x)$$

$= 5(\cos(e^x))^4(-\sin(e^x))\, e^x$

$\dfrac{d}{dx} = -5e^x(\cos(e^x))^4(\sin(e^x))$

Q4. [4 points] Find each of the following limits. You may use any technique or result we have seen so far in the course. Show your work.

0.5 (a) $\lim_{x\to\infty} \frac{x^2+\sqrt{x^6+x^4}}{2x-4x^3}$ Answer on back

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^6}+\sqrt{\frac{x^6}{x^6}+\frac{x^4}{x^6}}}{\frac{2x}{x^6}-\frac{4x^3}{x^6}}$ -1

$= \lim_{x\to\infty} \frac{\frac{1}{x^4}+\sqrt{1+\frac{1}{x^2}}}{\frac{2}{x^5}-\frac{4}{x^3}}$

$= \frac{0+\sqrt{1+0}}{0-0}$

$\therefore \lim_{x\to\infty} \frac{x^2+\sqrt{x^6+x^4}}{2x-4x^3} = 1$

-0.5 $\frac{1}{0} \neq 1$

1.5 (b) $\lim_{x\to 0} \frac{(x^2-4x)\sin(x)}{x^2}$

$= \lim_{x\to 0} \frac{x(x-4)\sin(x)}{x^2}$

$= \lim_{x\to 0}(x-4) \cdot \lim_{x\to 0} \frac{\sin(x)}{x}$

~~How~~ justify

-0.5 State $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$= (0-4)\cdot 1$

$= -4$

$\therefore \lim_{x\to 0} \frac{(x^2-4x)\sin(x)}{x^2} = -4$

$2x(1-2x^2)$

$\sqrt{x^6+x^4}$

$\frac{\frac{x^2}{x^3}+\frac{\sqrt{\;}}{x^3}}{\frac{2x}{x^3}-\frac{4x^3}{x^3}}$

$= \frac{\sqrt{x^6+x^4}}{x^3}$

$= \frac{\sqrt{x^6+x^4}}{-4x^3}$

4.a) $\lim\limits_{x \to \infty} \dfrac{x^2 + \sqrt{x^6 + x^4}}{2x - 4x^3}$

$= \lim\limits_{x \to \infty} \dfrac{x^2 + \sqrt{x^6\left(1 + \frac{1}{x^2}\right)}}{2x - 4x^3}$

$= \lim\limits_{x \to \infty} \dfrac{x^2 + x^3\sqrt{1 + \frac{1}{x^2}}}{x^3\left(\frac{2}{x^2} - 4\right)}$

$= \lim\limits_{x \to \infty} \dfrac{\cancel{x^3}\left(\cancel{\frac{1}{x}} + \sqrt{1 + \cancel{\frac{1}{x^2}}}\right)}{\cancel{x^3}\left(\cancel{\frac{2}{x^2}} - 4\right)}$

$= \dfrac{1}{-4}$

$= -\dfrac{1}{4}$

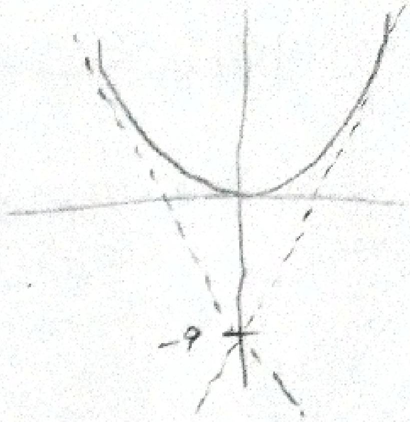
Q5. [2 points]

0.5

Find the equations of *all* lines which are tangent to $f(x) = x^2$ and pass through the point $(0, -9)$.

hint: Find the slope of the tangent line to $f(x)$ at $x = a$. Then find the equation of the tangent line to f at $x = a$. Then find the value(s) of a that make this line pass through the point $(0, -9)$.

$f(x) = x^2$ ① $(0, -9)$



① $f(a) = a^2$

$f'(a) = 2a$

② $y = mx + b$

$-9 = 2a(0) + b$

$-9 = b$

$2ax + b = y$ $2ax - 9 = y$

$2a(0) + b = -9$

$b = -9$

③ $y = 2ax - 9$

$-9 = 2a(0) - 9$

value of 'a' does not matter because it is being multiplied by 0.

$\therefore y = 2ax - 9$ where $\{a \in \mathbb{R}\}$ or $(-\infty, \infty)$

point on f(x) at $x = a$: $(a, f(a)) = (a, a^2)$

$y - y_0 = m(x - x_0)$

$y - a^2 = 2a(x - a)$

$(0, -9)$

$-9 - a^2 = 2a(0 - a)$

$-9 - a^2 = -2a^2$

$-9 = -a^2$

$\pm 3 = a$

$\therefore y - 9 = 6(x - 3)$

$y - 9 = -6(x + 3)$

$(f(x))^3$
$3(f(x))^2 f'(x)$

2

Q6. **[2 points]** Using implicit differentiation, find an expression for $\frac{dy}{dx}$ in terms of x and y, where x and y satisfy the following equation:

$$x^4y^3 = \tan(x) + 4x$$

$$4x^3y^3 + x^4 \cdot 3y^2 \cdot \frac{dy}{dx} = \sec^2(x) + 4$$

$$x^4 3y^2 \cdot \frac{dy}{dx} = \sec^2(x) + 4 - 4x^3y^3$$

$$\boxed{\frac{dy}{dx} = \frac{\sec^2(x) + 4 - 4x^3y^3}{3x^4y^2}}$$

2 **Q7.** **[2 points]** Consider the function $y = x^{(e^x)}$

Find an expression for y' purely in terms of x. *hint:* logarithmic differentiation

$$y = x^{(e^x)}$$

$$\ln y = \ln x^{(e^x)}$$

$$\ln y = e^x \ln x$$

$$\frac{1}{y} y' = e^x \ln x + e^x \cdot \frac{1}{x}$$

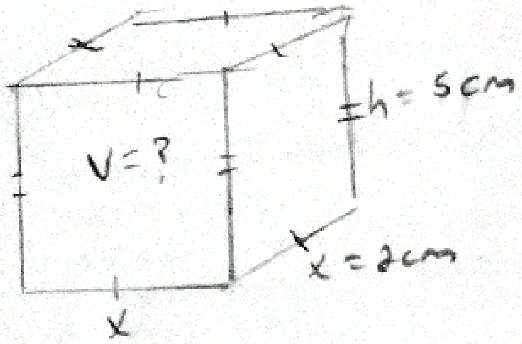
$$y' = \left(e^x \ln x + \frac{e^x}{x}\right) y$$

$$\boxed{y' = \left(e^x \ln x + \frac{e^x}{x}\right) x^{(e^x)}}$$

3

Q8. [**3 points**] A box with a square base is changing dimensions continuously. When the side of the base measures 2 cm and its height is 5 cm, the side of the base is growing at a rate of 1 cm/min while its height is decreasing at a rate of 3 cm/min. What is the rate of change of the box's volume at that moment in time? Clearly indicate what each variable in your solution represents and show all your steps.

let x be the sider of the base
let h be the height
let v be the volume



$x = 2cm$ $\qquad$ $h = 5cm$

$\frac{dx}{dt} = +1cm/min$ ← rate of x $\qquad$ $\frac{dh}{dt} = -3cm/min$ ← rate of h

$V = ?$ $\qquad$ $\frac{dV}{dt} = ?$ ← rate of V

① $V = x^2h$

$= (2cm)^2(5cm)$

$V = 20cm^3$

② Imp diff

$V = x^2h$

$\frac{dV}{dt} = (2x \cdot \frac{dx}{dt} \cdot h) + (x^2 \cdot \frac{dh}{dt})$

sub in known variables

$\frac{dV}{dt} = (2(2)(1)(5)) + (2^2 \cdot (-3))$

$= 20 + (-12)$

$\boxed{\frac{dV}{dt} = 8cm^3/min}$

∴ The volume of the box is increasing +8cm3/min.