Ryerson University

Department of Mathematics

Test 3

MTH 141 Linear Algebra

Student Number: ________________. Last Name:________________. First Name (**Print**):__________________.

Signature:______________________.

Professor (circle one)

C. Grandison	B. Tasić

Date: April 16, 2010, 11: 30 am

Duration: 1hr. 30 min.

Section : ________

Instructions:

1. Have your student card available on your desk.
2. This is a closed-book test. **Notes, calculators and other aids are not permitted.** Verify that your test has pages 1-8.
3. Do not separate the pages of this test booklet.
4. The point value of each question is indicated by the question number.
5. Include all significant steps in your solutions to the questions, presented in the correct order. **Unjustified answers will be given little or no credit.** Cross out or erase all rough work not relevant to your solution.
6. Present your solutions neatly and legibly in the space provided. **Messy or illegible solutions will receive no credit.**
7. If you need more space, use the back of the previous page. Indicate this fact on the original page.

For Instructor's use only.

Question	Mark
1	
2	
3	
4	
5	
Total	/50

1. [(5+3+2) marks] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,1) = (1,0,-1) \quad \text{and} \quad T(1,-1) = (3,-2,-5).$$

(a) Determine $T(x,y)$.

$T(1,1) + T(1,-1) = (1,0,-1) + (3,-2,-5) = (4,-2,-6) \Rightarrow$
$T((1,1)+(1,-1)) = (4,-2,-6) \Rightarrow T(2,0) = (4,-2,-6) \Rightarrow 2T(1,0) = (4,-2,-6) \Rightarrow$
$T(1,0) = (2,-1,-3) \quad \ldots\ (*)$
$T(1,1) - T(1,-1) = (1,0,-1) - (3,-2,-5) = (-2,2,4) \Rightarrow$
$T((1,1)-(1,-1)) = (-2,2,4) \Rightarrow T(0,2) = (-2,2,4) \Rightarrow 2T(0,1) = (-2,2,4) \Rightarrow$
$T(0,1) = (-1,1,2) \quad \ldots\ (**)$

Hence, the standard matrix of T is $[T] = [T(e_1)\ T(e_2)] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ -3 & 2 \end{bmatrix}$

and $T(x,y) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (2x-y, -x+y, -3x+2y)$

(b) Find the kernel ker(T).

$\mathrm{Ker}(T) = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0,0)\}$. Suppose $T(x,y) = (0,0,0)$.
Then $(2x-y, -x+y, -3x+2y) = (0,0,0)$.
Since the system $2x-y=0$, $-x+y=0$, $-3x+2y=0$ has only the trivial solution $x=y=0$ it follows $\mathrm{Ker}(T) = \{(0,0)\}$.

(c) Is T one-to one?

Since $\mathrm{Ker}(T) = \{(0,0)\}$ by Thm 6.3.11 it follows that T is one-to-one.

2. [(6+4) marks] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator that first rotates a vector about the x-axis through an angle θ, then projects the resulting vector orthogonally onto the xy-plane.

(a) Find the standard matrix for the linear operator T.

T can be expressed as the composition

$$T = T_B \circ T_A = T_{BA}$$

where A, B are the standard matrices for rotation and projection respectively. These matrices are:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence the standard matrix for T is

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Is the linear opeartor T onto ? Justify your answer.

$$T(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x, \; y\cos\theta - z\sin\theta \,; \; 0)$$

If $a \neq 0$ is an arbitrary real number, then $(0,0,a) \notin \operatorname{ran}(T)$ because there is no $(x,y,z) \in \mathbb{R}^3$ such that $T(x,y,z) = (0,0,a)$. So, T is not onto.

3. [(4+6) marks] Let t be a real parameter and $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$$T_t(x, y, z) = (x - y + 2z,\ x + 2y + tz,\ -y + z)$$

(a) Determine all values of t for which T is invertible.

T_t is invertible iff its standard matrix $[T_t] = [T(e_1) T(e_2) T(e_3)]$ is invertible iff $\det([T_t]) \neq 0$.

$$\det([T_t]) = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & t \\ 0 & -1 & 1 \end{vmatrix} = t + 1 \neq 0 \quad \text{iff} \quad t \neq -1$$

(b) Find $T_t^{-1}(x, y, z)$ for $t = 1$.

One finds $$[T_1]^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 & -5 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$T_1^{-1}(x_1, x_2, x_3) = \frac{1}{2}\begin{bmatrix} 3 & -1 & -5 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \frac{1}{2}(3x_1 - x_2 - 5x_3,\ -x_1 + x_2 + x_3,\ -x_1 + x_2 + 3x_3)$$

4. [(4+2+6) marks] Let $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (-1,1,0)$ and $\mathbf{u}_3 = (1,2,1)$ be vectors in $\mathbb{R}^3$ and let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(a) Show that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathbb{R}^3$. (**Hint:** Use determinants)

Solution: Since we have 3 vectors in $\mathbb{R}^3$ in order to show that B is a basis it suffices to show that B is linearly independent set. The set B is going to be linearly independent if and only if the only solution of the system:

$$x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 = \mathbf{0}$$

is trivial, i.e., $x = y = z = 0$. The above system is homogeneous and will have only the trivial solution if the determinant of the system is not zero. Computing the determinant of the above system one gets:

$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -1 \neq 0$. So, B is a basis for $\mathbb{R}^3$.

(b) Express the vector $\mathbf{w} = (1,2,3)$ as a linear combination of the vectors in the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We are looking for scalars x, y, z s.t

$$w = xu_1 + yu_2 + zu_3$$

This is equivalent to the system whose augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 0 & 1 & 2 \end{array}\right] \to \cdots\cdots \to \left[\begin{array}{ccc:c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The solution is $x = y = z = 1$ and hence

$$w = 6u_1 + 2u_2 - 3u_3.$$

(c) Let $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (-1,1,0)$ and $\mathbf{u}_3 = (1,2,1)$ be vectors in $\mathbb{R}^3$ and let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Use the Gram-Schmidt orthogonalization process to transform this basis into an orthogonal basis, and then normalize the orthogonal basis vectors to obtain an orthonormal basis for $\mathbb{R}^3$.

Solution: Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ denote the orthogonal basis produced by Gram-Schmidt orthogonalization process, and let $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ denote the orthonormal basis that results from normalizing $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$.

Step 2: Let $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2\cdot\mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 = (-1,1,0) - \frac{0}{3}(1,1,1) = (-1,1,0)$

Step 3: Let $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1,\mathbf{v}_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3\cdot\mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3\cdot\mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0) = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$.

Thus the vectors: $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (-1,1,0)$ and $\mathbf{v}_3 = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$ form an orthogonal basis for $\mathbb{R}^3$. The norms of these vectors are: $||\mathbf{v}_1|| = \sqrt{3}$, $||\mathbf{v}_2|| = \sqrt{2}$, and $||\mathbf{v}_3|| = \frac{\sqrt{6}}{6}$. So, an orthonormal basis for $\mathbb{R}^3$ is given by:

$\mathbf{w}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{||\mathbf{v}_2||} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\mathbf{w}_3 = \frac{\mathbf{v}_3}{||\mathbf{v}_3||} = (\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3})$.

5. [(5+3) marks]

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$.

(a) Given that $p_A(\lambda) = (\lambda - 2)^2(\lambda + 4)$ is the characteristic polynomial of the matrix A find the algebraic and geometric multiplicities of the eigenvalues of A.

Solution: The algebraic multiplicity of eigenvalue 2 is 2 and the algebraic multiplicity of eigenvalue -4 is 1. To determine geometric multiplicities of eigenvalues we need to find eigenspaces.

Eigenspace corresponding to the eigenvalue $\lambda = 2$ is the solution set of the system of equations $A\mathbf{x} = 2\mathbf{x}$ or equivalently $(A-2I_3)\mathbf{x} = \mathbf{0}$. The augmented matrix of the system $(A-2I_3)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 7 & -7 & 1 & 0 \\ 6 & -6 & 0 & 0 \end{bmatrix}$.

Reducing it to the RREF we get $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The corresponding equivalent system is: $x_1 - x_2 + x_3 = 0$ and $x_3 = 0$. Hence we get that the eigenspace is the set of all vectors $\mathbf{x} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, where $s \in \mathbb{R}$. This implies that the geometric multiplicity of eigenvalue 2 is 1.

Eigenspace corresponding to the eigenvalue $\lambda = -4$ is the solution set of the system of equations $A\mathbf{x} = -4\mathbf{x}$ or equivalently $(A + 4I_3)\mathbf{x} = \mathbf{0}$. The augmented matrix of the system $(A + 4I_3)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 7 & -1 & 1 & 0 \\ 7 & -1 & 1 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix}$. Reducing it to the RREF we get $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The corresponding equivalent system is: $x_1 - x_2 + x_3 = 0$ and $x_2 - x_3 = 0$. Hence we get that the eigenspace is the set of all vectors $\mathbf{x} = \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, where $s \in \mathbb{R}$. This implies that the geometric multiplicity of eigenvalue -4 is 1. Let us denote by: $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) Determine whether A is diagonalizable. If so, find a matrix P that diagonalizes the matrix A, and determine $P^{-1}AP$.

Solution: Since geometric multiplicity of eigenvalue 2 (which is 1) is not the same as algebraic multiplicity of eigenvalue 2 (which is 2) the matrix A is not diagonalizable by Thm 8.2.10.

Alternatively one can use:

Thm 8.2.9 and explain that A is not diagonalizable since the sum of the geometric multiplicities of its eigenvalues is 2, or

Thm 8.2.6 and explain that A is not diagonalizable since A has only 2 linearly independent eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$.