

Ryerson University
Department of Mathematics
MTH 141 Fall 2010 – Test III

LAST NAME: _____ FIRST NAME: _____
(Please print) (Please print)

I.D. NUMBER: _____ SIGNATURE: _____

Date: December 14, 2010

Duration: 90 minutes

Professor (circle one)

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Section: _____

INSTRUCTIONS:

- Verify that your test has **SIX (6)** pages including this page.
- The use of notes, formula sheets, books or calculators is not allowed.
- For full-answer questions:
Give full justification for your answers; correct answers alone may be worth nothing. Cross out or erase all rough work not relevant to your solution. Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there.
- For multiple choice questions:
Make sure to write your answers in the box at the end of each question carefully. There are no

part marks in the multiple-choice section and only the answer in the box will be marked. The correct response gets full marks, an incorrect response or no response gets no marks.

For markers' use only:

Page	Value	Mark
2	10	
3	10	
4	10	
5	10	
6	10	
Total	50	

1. (4+3+3 pts.) Suppose that $\lambda = 1$ is an eigenvalue of the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & -1 & 0 \\ m & -2 & 1 \end{bmatrix}$.

(a) Find m and determine the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$ of A .

As $\lambda = 1$ is an eigenvalue of A , $\det(I - A) = 0$

$$\begin{vmatrix} -3 & -2 & -1 \\ 0 & 2 & 0 \\ -m & 2 & 0 \end{vmatrix} = 0 \implies m = 0$$

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 4 & -2 & -1 \\ 0 & \lambda + 1 & 0 \\ 0 & 2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 1)(\lambda - 4) \end{aligned}$$

(b) Find the eigenspace corresponding to $\lambda = 1$.

We need to find the solution space of the system $(I - A)x = 0$.

$$I - A = \begin{bmatrix} -3 & -2 & -1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow[\substack{R_1 \rightarrow (-\frac{1}{3})R_1 \\ R_2 \rightarrow \frac{1}{2}R_2}]{R_3 \rightarrow R_3 - 2R_2} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = t$ ($t \in \mathbb{R}$). $x_1 = -\frac{1}{3}t$, $x_2 = 0$

So, the eigenspace is $\left\{ \left(-\frac{1}{3}t, 0, t\right) \mid t \in \mathbb{R} \right\}$

(c) Using $p(\lambda)$ obtained in (a), find all eigenvalues of the matrix $B = A - 3I$.

The characteristic polynomial of $B = A - 3I$ is

$$\begin{aligned} p^*(\lambda) &= \det(\lambda I - B) \\ &= \det(\lambda I - (A - 3I)) \\ &= \det((\lambda + 3)I - A) \\ &= p(\lambda + 3) \stackrel{\text{by (a)}}{=} (\lambda + 4)(\lambda + 2)(\lambda - 1) \end{aligned}$$

So, B has three eigenvalues: $\lambda_1 = -4$, $\lambda_2 = -2$, $\lambda_3 = 1$.

2. (2+2+3+3 pts.) Let $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be the linear transformation

$$T(x, y, z) = (x + y + 2z, 2x + 3y + 3z, -x - 2y - z).$$

(a) Find the standard matrix for T .

$$[T] = [T(e_1) \ T(e_2) \ T(e_3)] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ -1 & -2 & -1 \end{bmatrix}$$

(b) Is $w = (3, 4, -1)$ in the range of T ? Justify your answer.

$$(*) \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} \text{ has a solution } \iff w \in \text{ran}(T).$$

As $\begin{bmatrix} 1 & 1 & 2 & | & 3 \\ 2 & 3 & 3 & | & 4 \\ -1 & -2 & -1 & | & -1 \end{bmatrix} \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & | & 3 \\ 0 & 1 & -1 & | & -2 \\ 0 & -1 & 1 & | & 2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{bmatrix} 1 & 1 & 2 & | & 3 \\ 0 & 1 & -1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ From the last matrix, we know that (*) has solutions, so $w \in \text{ran}(T)$.

(c) Is T onto? Justify your answer.

As T is an operator, and T is onto $\iff T$ is one-to-one, by (d), we conclude that T is NOT onto.

(d) Show that T is not one-to-one.

$$T \text{ is one-to-one } \iff \text{Ker}(T) = \{0\}.$$

$$\text{As } \det([T]) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ -1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 0,$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ has non-trivial solutions.}$$

So, $\text{Ker}(T) \neq \{0\}$, thus T is not one-to-one.

3. (6+4 pts.)

(a) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the operator for which $T(1, 1) = (3, 2)$ and $T(0, -1) = (1, 1)$. Find the standard matrix for T , and then use it to find $T(2, 1)$.

$$T(e_1) = T[(1, 1) + (0, -1)] = T(1, 1) + T(0, -1) = (3, 2) + (1, 1) = (4, 3)$$

$$T(e_2) = T[-(0, -1)] = -T(0, -1) = (-1, -1)$$

$$\text{So, } [T] = [T(e_1) \ T(e_2)] = \begin{bmatrix} 4 & -1 \\ 3 & -1 \end{bmatrix}$$

$$T(2, 1) = \begin{bmatrix} 4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\text{or write } T(2, 1) = (7, 5)$$

(b) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the operator defined by

$$T(x, y) = (x - y, x + 4y).$$

Find $T^{-1}(x, y)$.

$$\text{The standard matrix for } T \text{ is } [T] = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$$

$$\text{The inverse of } [T] \text{ is } \frac{1}{5} \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\text{So, } T^{-1}(x, y) = \left(\frac{4}{5}x + \frac{1}{5}y, -\frac{1}{5}x + \frac{1}{5}y \right)$$

4. (5+5 pts.) Let $\mathbf{u}_1 = (1, 2, 1)$, $\mathbf{u}_2 = (-2, -1, -2)$ and $\mathbf{u}_3 = (4, 9, 5)$ be vectors in \mathbb{R}^3 .

(a) Show that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for \mathbb{R}^3 . (Hint: Use determinants)

$$\text{As } \begin{vmatrix} 1 & -2 & 4 \\ 2 & -1 & 9 \\ 1 & -2 & 5 \end{vmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ \hline R_3 \rightarrow R_3 - R_1 \end{array} \begin{vmatrix} 1 & -2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 * 3 * 1 = 3 \neq 0,$$

$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ form a basis for \mathbb{R}^3 .

(b) Express the vector $\mathbf{w} = (1, 1, 4)$ as a linear combination $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

Let $\mathbf{w} = x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3$. So

$$\begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 9 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 4 & | & 1 \\ 2 & -1 & 9 & | & 1 \\ 1 & -2 & 5 & | & 4 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \begin{bmatrix} 1 & -2 & 4 & | & 1 \\ 0 & 3 & 1 & | & -1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \begin{array}{l} R_2 \rightarrow \frac{1}{3}R_2 \end{array} \begin{bmatrix} 1 & -2 & 4 & | & 1 \\ 0 & 1 & \frac{1}{3} & | & -\frac{1}{3} \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - \frac{1}{3}R_3 \\ R_1 \rightarrow R_1 - 4R_3 \\ R_1 \rightarrow R_1 + 2R_2 \end{array} \begin{bmatrix} 1 & -2 & 4 & | & 1 \\ 0 & 1 & 0 & | & -\frac{4}{3} \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \begin{array}{l} R_1 \rightarrow R_1 - 4R_3 \\ R_1 \rightarrow R_1 + 2R_2 \end{array} \begin{bmatrix} 1 & -2 & 0 & | & -11 \\ 0 & 1 & 0 & | & -\frac{4}{3} \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \begin{array}{l} R_1 \rightarrow R_1 + 2R_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & | & -\frac{41}{3} \\ 0 & 1 & 0 & | & -\frac{4}{3} \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

So, $x = -\frac{41}{3}$, $y = -\frac{4}{3}$, $z = 3$, and thus

$$\mathbf{w} = -\frac{41}{3}\mathbf{u}_1 - \frac{4}{3}\mathbf{u}_2 + 3\mathbf{u}_3$$

5. (3+3+4 pts.) Let $A = \begin{bmatrix} -7 & 12 \\ -4 & 7 \end{bmatrix}$, and let $X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(a) Show that X_1, X_2 are eigenvectors of A and find the corresponding eigenvalues.

$$AX_1 = \begin{bmatrix} -7 & 12 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -X_1, \text{ so } X_1 \text{ is an eigenvector, and the corresponding eigenvalue } \lambda_1 = -1.$$

$$AX_2 = \begin{bmatrix} -7 & 12 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = X_2, \text{ so } X_2 \text{ is also an eigenvector, and the corresponding eigenvalue } \lambda_2 = 1$$

(b) Determine whether A is diagonalizable. If so, find a matrix P that diagonalizes the matrix A .

Since A has 2 distinct eigenvalues $\lambda_1 = -1$ & $\lambda_2 = 1$, A is diagonalizable.

$$\text{Take } P = [X_1 \ X_2] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \text{ and } P^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) Using (b) to compute A^{2010} .

$$\text{By (b), } A = P \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}.$$

$$A^{2010} = \left(P \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} \right)^{2010}$$

$$= P \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{2010} P^{-1}$$

$$= P \begin{bmatrix} (-1)^{2010} & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = P I P^{-1} = I$$