Econ 327 (001)
Winter Session Term I, 2018
M. Vaney

Problem Set 3 - Solutions

1. For each of the following functions,
   (i) find the constant c so that $f(x)$ is a p.d.f. of a random variable X,
   (ii) find the distribution function, $F(x) = \mathbb{P}(X \leq x)$,
   (iii) sketch graphs of the p.d.f. $f(x)$ and the distribution function $F(x)$,
   (iv) find μ_X,
   (v) find σ_X^2 and σ_X

(a)

$$f(x) = \frac{x^3}{4} \qquad 0 \leq x \leq c$$

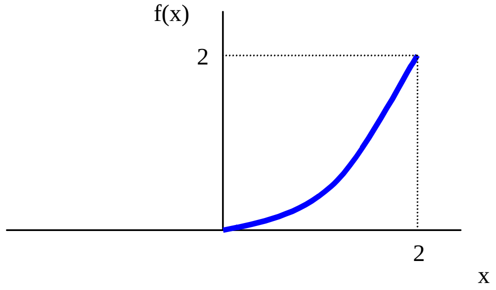
(i)

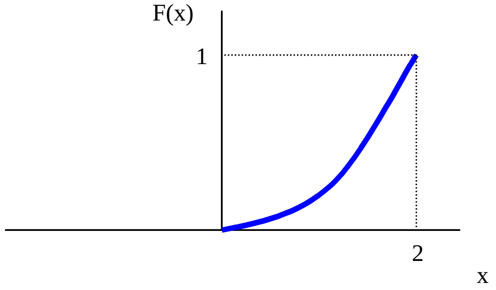
$$\begin{aligned} \int_0^c \frac{x^3}{4} dx &= 1 \\ \left.\frac{x^4}{16}\right]_0^c &= 1 \\ \frac{c^4}{16} - 0 &= 1 \\ c^4 &= 16 \\ c &= 2 \end{aligned}$$

(ii)

$$F(x) = \frac{x^4}{16} \qquad 0 < x < 2$$

(iii)





(iv)

$$\begin{aligned} \mu &= \int_0^2 x \frac{x^3}{4} dx \\ &= \left.\frac{x^5}{20}\right]_0^2 = \frac{8}{5} \end{aligned}$$

$$\begin{aligned}\sigma^2 &= \int_0^2 \left(x-\frac{8}{5}\right)^2 \frac{x^3}{4}dx \\ &= \int_0^2 \left(\frac{x^5}{4}-\frac{4}{5}x^4+\frac{16}{25}x^3\right)dx \\ &= \left(\frac{x^6}{24}-\frac{4x^5}{25}+\frac{4x^4}{25}\right)\Bigg]_0^2 \\ &= .1067 \\ \sigma &= .3266\end{aligned}$$

(b)

$$f(x)=\frac{3}{16}x^2 \qquad -c\le x\le c$$

(i)

$$\begin{aligned}\int_{-c}^{c}\frac{3}{16}x^2dx &= 1 \\ \frac{3x^3}{48}\Bigg]_{-c}^{c} &= 1 \\ \frac{3}{48}c^3-\frac{3}{48}(-c)^3 &= 1 \\ 2c^3 &= 16 \\ c &= 2\end{aligned}$$

$$f(x)=\frac{3}{16}x^2 \qquad -2\le x\le 2$$

(ii)

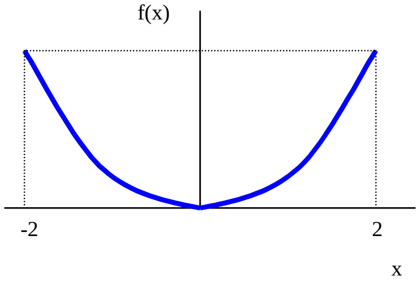
$$\begin{aligned}F(-2) &= 0 \\ F(-2) &= \frac{3}{48}(-2)^3+k=0 \\ k &= \frac{1}{2}\end{aligned}$$

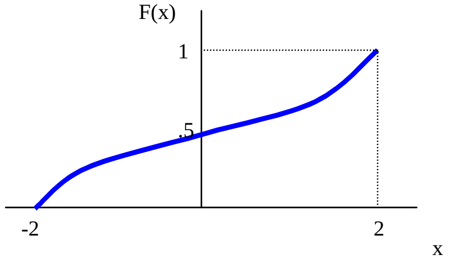
or

$$\begin{aligned}F(2) &= 1 \\ F(2) &= \frac{3}{48}(2)^3+k=1 \\ k &= \frac{1}{2}\end{aligned}$$

$$F(x)=\frac{3}{48}x^3+\frac{1}{2} \qquad -2\le x\le 2$$

(iii)





(iv)

$$\begin{aligned}
\mu &= \int_{-2}^{2} \frac{3}{16} x^3 dx \\
&= \frac{3}{64} x^4 \Big]_{-2}^{2} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \int_{-2}^{2} \frac{3}{16} x^4 dx \\
&= \frac{3}{80} x^5 \Big]_{-2}^{2} \\
&= \frac{12}{5} \\
\sigma &= 1.55
\end{aligned}$$

(c)

$$f(x) = \frac{c}{\sqrt{x}} = cx^{-\frac{1}{2}} \qquad 0 \leq x \leq 1$$

(i)

$$\begin{aligned}
\int_0^1 cx^{-\frac{1}{2}} dx &= 1 \\
2cx^{\frac{1}{2}} \Big]_0^1 &= 1 \\
2c - 0 &= 1 \\
c &= \frac{1}{2}
\end{aligned}$$

$$f(x) = \frac{1}{2} x^{-\frac{1}{2}} \qquad 0 \leq x \leq 1$$

we can note that

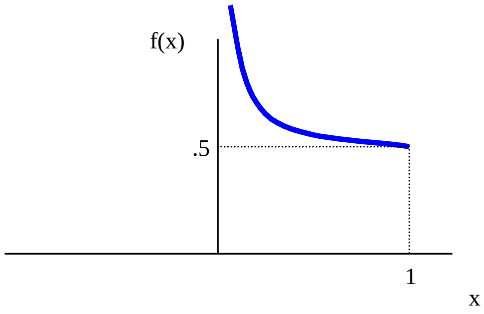
$$\lim_{x \to 0} f(x) = \infty$$

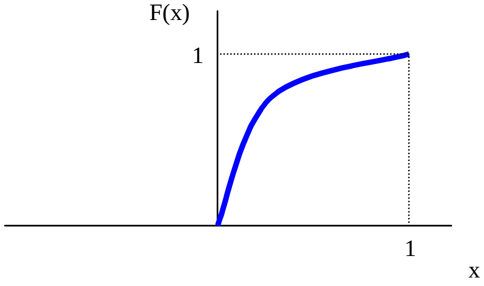
so the pdf is unbounded as x approaches 0 (the density is finite at $f(1)$)

(ii)

$$F(x) = x^{\frac{1}{2}} \qquad 0 \le x \le 1$$

(iii)





(iv)

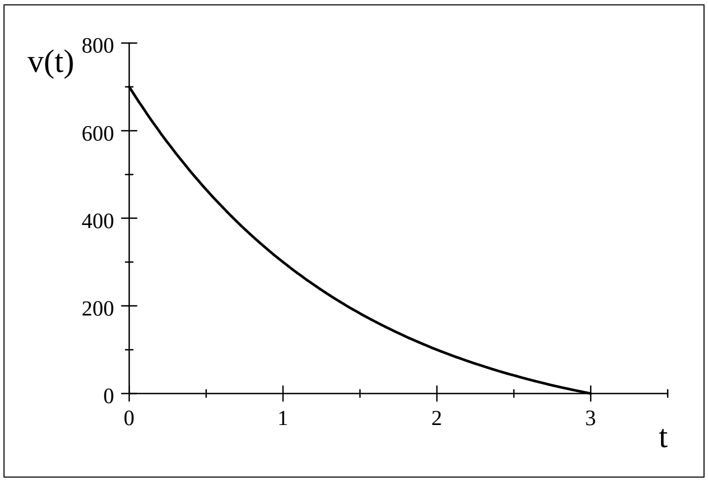
$$\begin{aligned}
\mu &= \int_0^1 \frac{x^{\frac{1}{2}}}{2} dx \\
&= \left.\frac{x^{\frac{3}{2}}}{3}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \int_0^1 \left(x - \frac{1}{3}\right)^2 \frac{1}{2x^{\frac{1}{2}}} dx \\
&= \int_0^1 \left(\frac{1}{2}x^{\frac{3}{2}} - \frac{2}{6}x^{\frac{1}{2}} + \frac{1}{18}x^{-\frac{1}{2}}\right) dx \\
&= \left.\frac{x^{\frac{5}{2}}}{5} - \frac{2x^{\frac{3}{2}}}{9} + \frac{x^{\frac{1}{2}}}{9}\right]_0^1 \\
&= \frac{4}{45} \\
\sigma &= .2981
\end{aligned}$$

2. The initial value of an appliance is \$700 and its value in the future, for purposes of the warranty, is given by:

$$v(t) = 100(2^{3-t} - 1), \quad 0 \le t \le 3$$

where t is time in years. The appliance has a three year warranty that pays $v(t)$.



(a) the value of the appliance at the end of 1 year will be:

$$v(1) = 100(2^{3-1} - 1) = \$300$$

(b) the value of the appliance at the end of 3 years will be:

$$v(3) = 100(2^{3-3} - 1) = 0$$

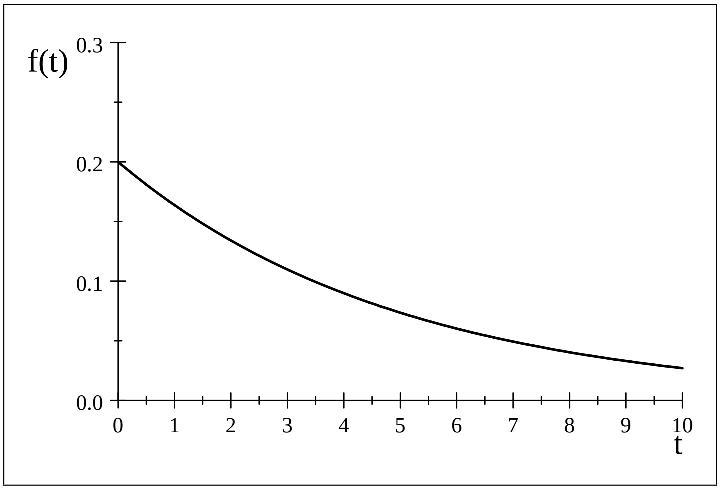
(c) The value of appliance at time t is given by:

$$v(t) = 100(2^{3-t} - 1) \qquad 0 \leq t \leq 3$$

The value of the warranty ranges from \$700 if failure occurs at $t = 0$ down to \$0 if failure occurs at the expiry of the warranty at $t = 3$.

The time to failure of the appliance, T, is distributed as exponential with mean $\theta = 5$

$$\begin{aligned} f(t) &= \frac{1}{\theta} e^{\frac{-t}{\theta}} \\ &= \frac{1}{5} e^{\frac{-t}{5}} \qquad 0 \leq t \leq \infty \end{aligned}$$



For the expected value of the payment on the warranty we find:

$$\begin{aligned}
\mathbb{E}\left\{v(t)\right\} &= \int_0^3 v(t)f(t)dt \\
&= \int_0^3 100(2^{3-t}-1)\frac{1}{5}e^{\frac{-t}{5}}dt \\
&= 20\int_0^3 (2^{3-t}-1)e^{\frac{-t}{5}}dt \\
&= 20\int_0^3 2^{3-t}e^{\frac{-t}{5}}dt - 20\int_0^3 e^{\frac{-t}{5}}dt
\end{aligned}$$

noting that $x^a = e^{a\ln x}$ we can write $2^{3-t} = \left(2^3\right)\left(2^{-t}\right) = 8(2^{-t}) = 8e^{-t\ln 2}$

$$\begin{aligned}
&= 20\int_0^3 8e^{-t\ln 2}e^{\frac{-t}{5}}dt - 20\int_0^3 e^{\frac{-t}{5}}dt \\
&= 160\int_0^3 e^{-t\ln 2-\frac{t}{5}}dt - 20\int_0^3 e^{\frac{-t}{5}}dt \\
&= 160\int_0^3 e^{-t(\ln 2+.2)}dt - 20\int_0^3 e^{\frac{-t}{5}}dt \\
&= 160\left(-\frac{1}{\ln 2+.2}e^{-t(\ln 2+.2)}\Big]_0^3\right) - 20\left(-5e^{\frac{-t}{5}}\Big]_0^3\right) \\
&= -179.136\left(e^{-t(\ln 2+.2)}\Big]_0^3\right) + 100\left(e^{\frac{-t}{5}}\Big]_0^3\right)
\end{aligned}$$

evaluating this we have

$$\begin{aligned}
-179.136\left(e^{-t(\ln 2+.2)}\Big]_0^3\right) &= 166.85 \\
100\left(e^{\frac{-t}{5}}\Big]_0^3\right) &= -45.11
\end{aligned}$$

$$\mathbb{E}\left\{v(t)\right\} = 166.85 - 45.11 = \$121.74$$

3. $X \sim N(6, 25)$. Probabilities associated with intervals under this distribution can be found by converting to equivalent standard normal values $Z = \frac{X-6}{5}$.

(a) $\mathbb{P}\left(6 \le X \le 12\right)$

$$\begin{aligned}
\mathbb{P}\left(6 \le X \le 12\right) &= \mathbb{P}\left(0 \le Z \le 1.2\right) \\
&= \Phi(1.2) - \Phi(0) \\
&= .8849 - .5 = .3849
\end{aligned}$$

(b) $\mathbb{P}\left(X > 21\right)$

$$\begin{aligned}
\mathbb{P}\left(X > 21\right) &= \mathbb{P}\left(Z > \frac{21-6}{5}\right) \\
&= \mathbb{P}\left(Z > 3\right) \\
&= 1 - \Phi(3) = .0013
\end{aligned}$$

(c) $\mathbb{P}\left(|X-6|<5\right)$

$$\begin{aligned}
\mathbb{P}\left(|X-6|<5\right) &= \mathbb{P}\left(-5 \le X-6 \le 5\right) \\
&= \mathbb{P}\left(1 \le X \le 11\right) \\
&= \mathbb{P}\left(\frac{1-6}{5} \le Z \le \frac{11-6}{5}\right) \\
&= \mathbb{P}\left(-1 \le Z \le 1\right) \\
&= \Phi(1) - \Phi(-1) \\
&= .8413 - .1587 \\
&= .6826
\end{aligned}$$

4. $X \sim N(21.37, .16)$

(a)

$$\begin{aligned}
\mathbb{P}(X &> 22.07) \\
&= \mathbb{P}(Z > \frac{22.07-21.37}{.4}) \\
&= \mathbb{P}(Z > 1.75) \\
&= 1 - \mathbb{P}(Z < 1.75) \\
&= .0401
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{P}(X &< 20.857) \\
&= \mathbb{P}(Z < \frac{20.857-21.37}{.4}) \\
&= \mathbb{P}(Z < -1.2825) \\
&= .1
\end{aligned}$$

Y is counting the number of successes (candies below a threshold weight) in a fixed number of trials (15). Y will be binomial: $Y \sim b(15, .1)$

$$\mathbb{P}(Y \le 2) = f(0) + f(1) + f(2)$$

$$\begin{aligned}
f(0) &= \binom{15}{0}(.1)^0(.9)^{15} = .2059 \\
f(1) &= \binom{15}{1}(.1)^1(.9)^{14} = .3432 \\
f(2) &= \binom{15}{2}(.1)^2(.9)^{13} = .2669
\end{aligned}$$

$$\mathbb{P}(Y \le 2) = .816$$

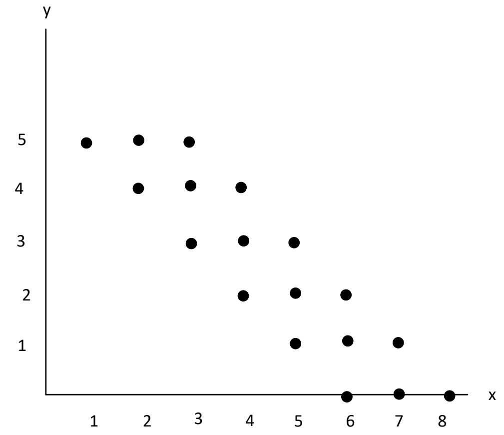
5. For each of the following functions, determine the constant c so that $f(x,y)$ satisfies the conditions of being a joint pmf for two discrete random variables X and Y.

(a) $f(x,y) = c(x+2y) \qquad x = 1,2 \text{ and } y = 1,2,3$

$$\begin{aligned}
\sum_{x=1}^{2}\sum_{y=1}^{3} c(x+2y) &= 1 \\
c\sum_{x=1}^{2}\sum_{y=1}^{3} (x+2y) &= 1 \\
33c &= 1 \\
c &= \frac{1}{33}
\end{aligned}$$

(b) $f(x,y) = c \qquad x$ and y are integers such that $6 \le x+y \le 8, \qquad 0 \le y \le 5$

The sample space can be represented:



All points in the sample space occur with equal probability. $c = \frac{1}{18}$

Note that the marginals in this case will be $f_2(y) = \frac{1}{6}\ y = 0,1,\ldots,6$.

For

$$f_1(x) = \begin{cases} \frac{1}{18} & x = 1,8 \\ \frac{1}{9} & x = 2,7 \\ \frac{1}{6} & x = 3,4,5,6 \end{cases}$$

6. Let the joint pmf of X and Y be defined by $f(x,y) = \frac{x+y}{32}, \qquad x = 1,2 \text{ and } y = 1,2,3,4$

(a) Find $f_x(x)$, the marginal pmf of X.

$$\begin{aligned}
f_x(x) &= \sum_{y=1}^{4} f(x,y) \\
&= \sum_{y=1}^{4} \frac{x+y}{32} \\
&= \frac{1}{8}x + \frac{5}{16}
\end{aligned}$$

(b) Find $f_y(y)$, the marginal pmf of Y.

$$\begin{aligned} f_y(y) &= \sum_{x=1}^{2} f(x,y) \\ &= \sum_{x=1}^{2} \frac{x+y}{32} \\ &= \frac{1}{16}y + \frac{3}{32} \end{aligned}$$

(c) Find $\mathbb{P}(X > Y)$

There is only one outcome with $X > Y : X = 2, Y = 1$

$$\begin{aligned} \mathbb{P}(X > Y) &= \mathbb{P}(X = 2, Y = 1) \\ &= f(2,1) \\ &= \frac{2+1}{32} = \frac{3}{32} \end{aligned}$$

(d) Find $\mathbb{P}(Y = 2X)$

$$\begin{aligned} \mathbb{P}(Y = 2X) &= \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 4) \\ &= f(1,2) + f(2,4) \\ &= \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32} \end{aligned}$$

(e) Find $\mathbb{P}(X + Y = 3)$

$$\begin{aligned} & \mathbb{P}(X + Y = 3) \\ =\ & \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 1) \\ =\ & \frac{1+2}{32} + \frac{2+1}{32} = \frac{3}{16} \end{aligned}$$

(f)

$$\begin{aligned} f_x(x) f_y(y) &= \left(\frac{1}{8}x + \frac{5}{16}\right)\left(\frac{1}{16}y + \frac{3}{32}\right) \\ &= \frac{3}{256}x + \frac{5}{256}y + \frac{1}{128}xy + \frac{15}{512} \\ &\neq f(x,y) = \frac{x+y}{32} \end{aligned}$$

(g) Find the mean and variance of X and Y.

$$\begin{aligned} \mu_X &= \mathbb{E}\{X\} = \sum_{x=1}^{2}\sum_{y=1}^{4} x f(x,y) = \sum_{x=1}^{2}\sum_{y=1}^{4} x \frac{x+y}{32} \\ &= \sum_{x=1}^{2}\sum_{y=1}^{4} \frac{x^2 + xy}{32} \\ &= \frac{25}{16} \end{aligned}$$

We can also use the marginal p.m.f. for x to find μ_X:

$$\begin{aligned}
\mu_X &= \mathbb{E}\{X\} = \sum_{x=1}^{2} x f_x(x) \\
&= \sum_{x=1}^{2} x\left(\frac{1}{8}x + \frac{5}{16}\right) \\
&= \sum_{x=1}^{2} \left(\frac{1}{8}x^2 + \frac{5}{16}x\right) \\
&= \frac{25}{16}
\end{aligned}$$

In general, consider the mean of X. We have

$$\mathbb{E}\{X\} = \sum_x \sum_y x f(x,y)$$

and

$$\mathbb{E}\{X\} = \sum_x x f_x(x)$$

To show that these are equivalent we can note that x does not change as y changes so x can be taken through the y summation:

$$\sum_x \sum_y x f(x,y) = \sum_x x \sum_y f(x,y)$$

and summing $f(x,y)$ over all y (holding x fixed) will give the marginal for x

$$\sum_x x \sum_y f(x,y) = \sum_x x f_x(x)$$

and we have equivalence:

$$\mathbb{E}\{X\} = \sum_x \sum_y x f(x,y) = \sum_x x f_x(x)$$

For the mean of Y :

$$\begin{aligned}
\mathbb{E}\{Y\} &= \sum_x \sum_y y f(x,y) \\
&= \sum_{x=1}^{2} \sum_{y=1}^{4} y \frac{x+y}{32} \\
&= \frac{45}{16}
\end{aligned}$$

Variance of X :

$$\begin{aligned}\mathbb{E}\left\{X^2\right\} &= \sum_{x=1}^{2}\sum_{y=1}^{4} x^2\frac{x+y}{32} \\ &= \frac{43}{16}\end{aligned}$$

$$\begin{aligned}\sigma_X^2 &= \mathbb{E}\left\{X^2\right\} - \mu_X^2 \\ &= \frac{43}{16} - (\frac{25}{16})^2 \\ &= \frac{63}{256}\end{aligned}$$

and variance of Y

$$\begin{aligned}\sigma_Y^2 &= \mathbb{E}\left\{Y^2\right\} - \mu_Y^2 \\ &= \sum_{x=1}^{2}\sum_{y=1}^{4} y^2\frac{x+y}{32} - \left(\frac{45}{16}\right)^2 \\ &= \frac{145}{16} - \frac{2025}{256} \\ &= \frac{295}{256}\end{aligned}$$

and covariance

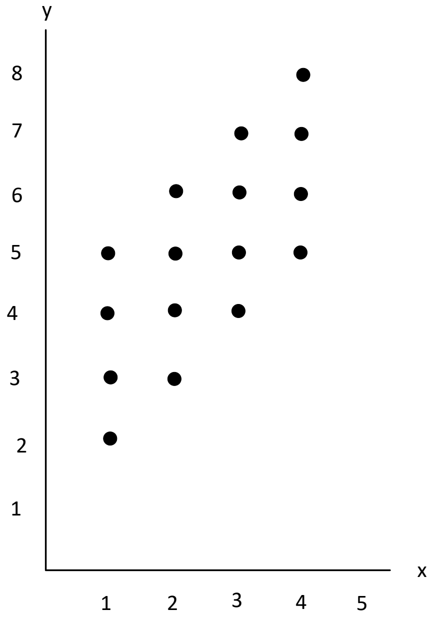
$$\begin{aligned}\mathbb{E}\left\{XY\right\} &= \sum_{x=1}^{2}\sum_{y=1}^{4} xy\frac{x+y}{32} \\ &= \frac{35}{8}\end{aligned}$$

$$\begin{aligned}\sigma_{XY} &= \mathbb{E}\left\{XY\right\} - \mathbb{E}\left\{X\right\}\mathbb{E}\left\{Y\right\} \\ &= \frac{35}{8} - \left(\frac{25}{16}\right)\left(\frac{45}{16}\right) \\ &= -\frac{5}{256}\end{aligned}$$

and the correlation coefficient:

$$\begin{aligned}\rho_{XY} &= \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \\ &= \frac{-\frac{5}{256}}{\left(\frac{63}{256}\right)^{\frac{1}{2}}\left(\frac{295}{256}\right)^{\frac{1}{2}}} \\ &= -.00367\end{aligned}$$

7. Roll a fair four-sided die twice. Let X equal the outcome on the first roll, and let Y equal the sum of the two rolls. $x = 1, 2, 3, 4$ and $y = 2, 3, 4, \ldots, 8$. The sample space can be represented by:



All points in the sample space occur with equal probability so the joint p.m.f. will be $f(x, y) = \frac{1}{16}$ for all (x, y) in the sample space

(a) μ_X. Note that $f_1(x) = \frac{1}{4}$ for $x = 1, 2, 3, 4$. $\mu_X = \sum_{x=1}^{4} x\frac{1}{4} = 2.5$

$\mu_Y = 2(\frac{1}{16}) + 3(\frac{1}{8}) + 4(\frac{3}{16}) + 5(\frac{1}{4}) + 6(\frac{3}{16}) + 7(\frac{1}{8}) + 8(\frac{1}{16}) = 5$

$\sigma_X^2 = \mathbb{E}\{X^2\} - \mu_X^2 = \sum_{x=1}^{4} x^2\frac{1}{4} - (2.5)^2 = 7.5 - 6.25 = 1.25$

$\sigma_Y^2 = \mathbb{E}\{Y^2\} - \mu_Y^2 = 4(\frac{1}{16}) + 9(\frac{1}{8}) + 16(\frac{3}{16}) + 25(\frac{1}{4}) + 36(\frac{3}{16}) + 49(\frac{1}{8}) + 64(\frac{1}{16}) - 5^2 = :$
$27.5 - 25 = 2.5$

$Cov(X, Y) = \mathbb{E}\{XY\} - \mu_X\mu_Y$

$= \frac{1}{16}(2+3+4+5+6+8+10+12+12+15+18+21+20+24+28+32) - (2.5)(5)$

$= 13.5 - 12.5 = 1$

$\rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{1}{\sqrt{1.25}\sqrt{2.5}} = 0.57$

8. A fair, six-sided die is rolled 5 independent times. Let X be the number of ones and Y the number of twos.

(a) The joint p.m.f. will be described by the trinomial distribution, Let $p_x = \frac{1}{6}$, $p_y = \frac{1}{6}$ and $1 - p_x - p_y = \frac{2}{3}$ be the probability of a roll different from 1 or 2. Let $n = 5$.

$$f(x, y) = \frac{5!}{x!y!(5-x-y)!}\left(\frac{1}{6}\right)^x\left(\frac{1}{6}\right)^y\left(\frac{2}{3}\right)^{5-x-y}$$

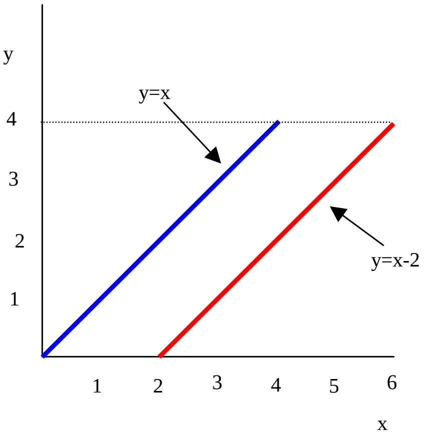
(b) Given $X = x$ then the conditional p.m.f. of Y can take values from $y = 0, 1, \ldots, n-x$. For example, conditional on $x = 3$ (in 5 rolls of the die the number of times that a 'one' is rolled is 3) then there can be at most $5 - 3 = 2$ rolls with a 'two'. The conditional probability of rolling a 'two' will be $\frac{p_y}{1-p_x}$: any roll that is not a 'one' can be a 'two' or something else (three or higher). The conditional distribution of Y will be binomial, $b(n-x, \frac{p_y}{1-p_x}) = b(5-x, \frac{1}{5})$.

(c) Since the conditional distribution of Y given a realization of $X = x$ is binomial the conditonal mean will be

$$\begin{aligned}
\mathbb{E}\{Y|x\} &= (5-x)\left(\frac{p_y}{1-p_x}\right) \\
&= (5-x)\frac{1}{5} \\
&= 1 - \frac{1}{5}x
\end{aligned}$$

9. Let $f(x,y) = \frac{1}{8}, \quad 0 \le y \le 4, \quad y \le x \le y+2$ be the joint *pdf* of X and Y.

(a) We see with the description of the joint p.d.f. that y can take values in the interval $(0,4)$ and x can take values in the interval $(0,6)$. The sample space (region where $f(x,y) > 0$) will be the region between $y = x$ and $y = x - 2$ with $0 < y < 4$.



(b) for the marginal p.d.f. of x, visual inspection of the sample space reveals that this will not be a constant. Given the change in the vertical height of the sample space we can see that the marginal p.d.f. of x is increasing over $0 \le x \le 2$, constant over $2 \le x \le 4$ and decreasing over $4 \le x \le 6$.

$$f_1(x) = \int_y f(x,y)dy$$

for $0 \le x \le 2$, y can take values from 0 up to x (since y has to be less than x)

$$f_1(x) = \int_0^x \frac{1}{8}dy = \frac{1}{8}y\Big]_0^x = \frac{1}{8}x \qquad 0 \le x \le 2$$

for $2 \le x \le 4$, y can take any values from $x-2$ up to x

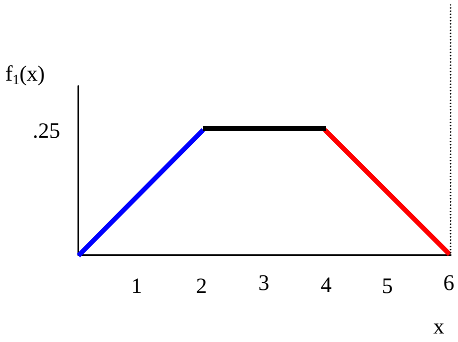
$$f_1(x) = \int_{x-2}^{x} \frac{1}{8} dy = \frac{1}{8} y \Big]_{x-2}^{x} = \frac{1}{8}x - \frac{1}{8}x + \frac{1}{4} = \frac{1}{4} \qquad 2 \le x \le 4$$

for $4 \le x \le 6$, y takes values from $x-2$ up to 4

$$f_1(x) = \int_{x-2}^{4} \frac{1}{8} dy = \frac{1}{8} y \Big]_{x-2}^{4} = \frac{1}{2} - \frac{1}{8}x + \frac{1}{4} = \frac{3}{4} - \frac{1}{8}x \qquad 4 \le x \le 6$$

we can write the marginal p.d.f. of x as

$$f_1(x) = \begin{cases} \frac{1}{8}x & 0 \le x \le 2 \\ \frac{1}{4} & 2 \le x \le 4 \\ \frac{3}{4} - \frac{1}{8}x & 4 \le x \le 6 \end{cases}$$



(c) for the marginal p.d.f. of y, visual inspection of the sample space reveals that this should be constant over $0 \le y \le 4$.

$$\begin{aligned} f_2(y) &= \int_{y}^{y+2} \frac{1}{8} dx = \frac{1}{8} x \Big]_{y}^{y+2} \\ &= \frac{1}{8}(y+2) - \frac{1}{8}y \\ &= \frac{1}{4} \qquad 0 \le y \le 4 \end{aligned}$$

(d) conditional p.d.f. of Y given $X = x$

$$h(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$h(y|x) = \begin{cases} \frac{\frac{1}{8}}{\frac{x}{8}} = \frac{1}{x} & 0 \le y \le x \le 2 \\ \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2} & 2 \le x \le 4, \; x-2 \le y \le x \\ \frac{\frac{1}{8}}{\frac{6-x}{8}} = \frac{1}{6-x} & x-2 \le y \le 4 \le x \le 6 \end{cases}$$

(e)

$$g(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2} \qquad y \le x \le y+2$$

(f) note that given the conditions on x and y ($0 \leq y \leq 4$ and $y \leq x \leq y+2$) for a given value of x, y can take values over an interval that is at most of length 2. For instance, if $x = 1$ (or any value below 2) then y can only take values in the interval $(0, x)$ and we can find the conditional expectation as

$$\begin{aligned}
\mathbb{E}\{Y|x\} &= \int_0^x yh(y|x)dy \\
&= \int_0^x y\frac{1}{x}dy = \frac{1}{x}\int_0^x ydy \\
&= \frac{1}{x}\left(\frac{1}{2}y^2\Big]_0^x\right) \\
&= \frac{1}{x}\left(\frac{1}{2}x^2\right) \\
&= \frac{1}{2}x \qquad \text{when } 0 \leq x \leq 2
\end{aligned}$$

intuitively this should make sense since looking at $h(y|x)$ when $0 \leq x \leq 2$ we see that the conditional density of y is simply uniform over $(0, x)$ (density is $\frac{1}{x}$) and the expected value of a uniform random varible is the midpoint of the interval $\frac{1}{2}(a+b) = \frac{1}{2}x$.

when x takes a value in $2 \leq x \leq 4$ then y can take values anywhere from $x-2$ to x and

$$\begin{aligned}
\mathbb{E}\{Y|x\} &= \int_{x-2}^x yh(y|x)dy \\
&= \int_{x-2}^x y\frac{1}{2}dy \\
&= \frac{1}{4}y^2\Big]_{x-2}^x \\
&= \frac{1}{4}x^2 - \frac{1}{4}(x-2)^2 \\
&= \frac{1}{4}x^2 - \frac{1}{4}(x^2 - 4x + 4) \\
&= \frac{1}{4}x^2 - \frac{1}{4}x^2 + x - 1 \\
&= x - 1
\end{aligned}$$

intuitively this should also make sense, the conditional p.d.f. is again uniform over an interval of length 2 (from $x-2$ to x) so the expected value is half-way along this interval, i.e. $x-1$.

when x takes a value in $4 \leq x \leq 6$ then y takes values anywhere from $x-2$ up to 4. As for the previous cases, y will conditionally distributed as uniform over the permissible range. Take the case of $X = 6$. The only value that Y can take if $X = 6$ is $y = 4$ and we should expect the conditional expectation of y in this case to be 4 (you can see that $h(4|6)$ is undefined ($= \frac{1}{6-6}$) - the density is going to infinity at $y = 4$ since the distribution becomes a single point (trying to get area beneath a point equal to 1)

$$\begin{aligned}
\mathbb{E}\{Y|x\} &= \int_{x-2}^{4} y\frac{1}{6-x}dy \\
&= \frac{1}{6-x}\int_{x-2}^{4} ydy \\
&= \frac{1}{6-x}\left(\frac{1}{2}y^2\Big]_{x-2}^{4}\right) \\
&= \frac{1}{6-x}\left(8-\frac{1}{2}(x-2)^2\right) \\
&= \frac{1}{6-x}\left(8-\frac{1}{2}(x^2-4x+4)\right) \\
&= \frac{1}{6-x}(8-\frac{1}{2}x^2+2x-2) \\
&= \frac{1}{6-x}(6-\frac{1}{2}x^2+2x)
\end{aligned}$$

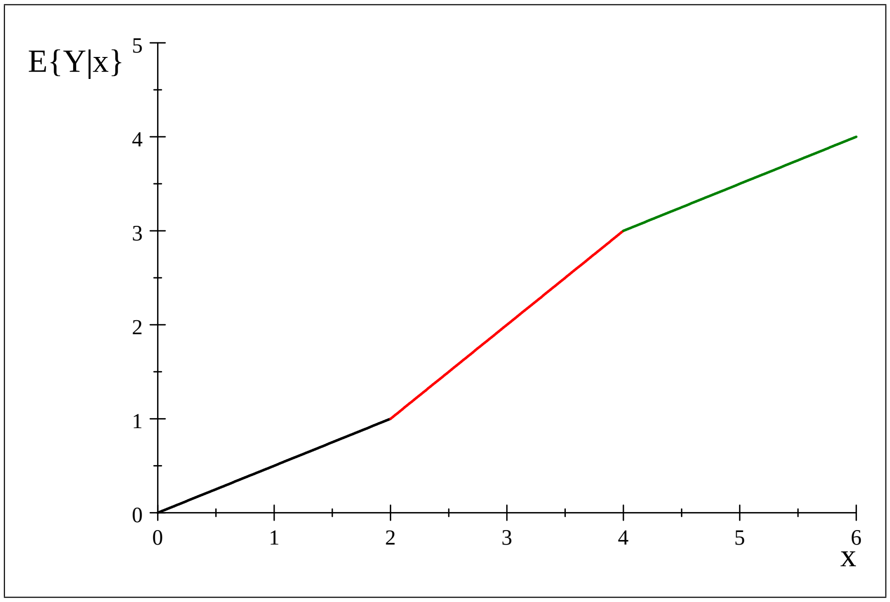
factor $(6-\frac{1}{2}x^2+2x)$ into $(6-x)(\frac{1}{2}x+1)$

$$\begin{aligned}
&= \frac{1}{6-x}(6-x)(\frac{1}{2}x+1) \\
&= \frac{1}{2}x+1
\end{aligned}$$

We have

$$\mathbb{E}\{Y|x\} = \begin{cases} \frac{1}{2}x & \text{when } 0 \le x \le 2 \\ x-1 & \text{when } 2 \le x \le 4 \\ \frac{1}{2}x+1 & \text{when } 4 \le x \le 6 \end{cases}$$

(g) The conditional mean of Y is a piecewise linear function of the realization of $X.y = \frac{1}{2}x$



10. Let X and Y have a bivariate normal distribution. with $\mu_X = 70$, $\sigma_X^2 = 100$, $\mu_Y = 80, \sigma_Y^2 = 169$ and $\rho = \frac{5}{13}$. Find:

(a) $\mathbb{E}\{Y|X = 72\}$

$$\begin{aligned} \mathbb{E}\{Y|X = x\} &= \mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(x - \mu_X) \\ &= 80 + \left(\frac{5}{13}\right)\frac{13}{10}(72 - 70) \\ &= 81 \end{aligned}$$

(b) $\mathrm{Var}(Y|X = 72)$

$$\begin{aligned} var(Y|X &= 72) = \sigma^2_{Y|_{X=72}} = \sigma_Y^2(1 - \rho_{XY}^2) \\ &= 169(1 - \frac{5}{13}^2) \\ &= 144 \end{aligned}$$

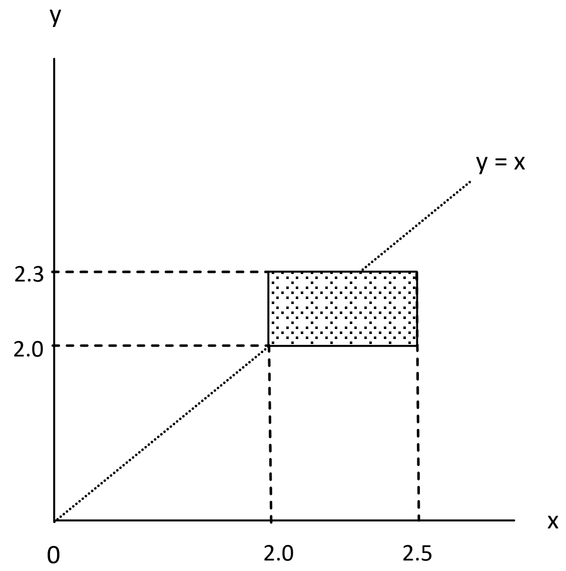
(c) $\mathbb{P}(Y \leq 84|X = 72)$

$$Y|_{X=72} \sim N(81, 144)$$

$$\begin{aligned} \mathbb{P}(Y \leq 84|X = 72) &= \mathbb{P}\left(\frac{Y - \mu_{Y_{|X=72}}}{\sigma_{Y|_{X=72}}} \leq \frac{84 - 81}{12}\right) \\ &= \mathbb{P}\left(Z \leq \frac{3}{12}\right) \\ &= \Phi(.25) \\ &= .5987 \end{aligned}$$

11. Two construction companies make bids of X and Y on a project. The joint pdf of X and Y is bivariate uniform on the space $2 \leq x \leq 2.5$ and $2 \leq y \leq 2.3$. If X and Y are within 0.1 of each other the companies will be asked to rebid, otherwise, the low bidder will be awarded the contract.

(a)



(b) Joint pdf $f(x, y)$:

$$\int_2^{2.5} \int_2^{2.3} f(x, y) dy dx = 1$$

With X and Y jointly uniform (how do you know the two random variables are independent?), $f(x, y) = c$

$$\begin{aligned}
\int_2^{2.5} \int_2^{2.3} c dy dx &= 1 \\
c \int_2^{2.5} \int_2^{2.3} dy dx &= 1 \\
c \int_2^{2.5} \int_2^{2.3} dy dx &= 1 \\
c \int_2^{2.5} \left(y]_2^{2.3} \right) dx &= 1 \\
c \int_2^{2.5} .3 dx &= 1 \\
c \int_2^{2.5} dx &= \frac{10}{3} \\
c \left(x]_2^{2.5} \right) &= \frac{10}{3} \\
c &= \frac{20}{3}
\end{aligned}$$

$$f(x, y) = \begin{cases} \frac{20}{3} & \text{for } x, y \text{ with } 2 \leq x \leq 2.5 \text{ and } 2 \leq y \leq 2.3 \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
f_x(x) &= \int_2^{2.3} \frac{20}{3} dy \\
f_x(x) &= 2 \text{ for } 2 \leq x \leq 2.5
\end{aligned}
$$

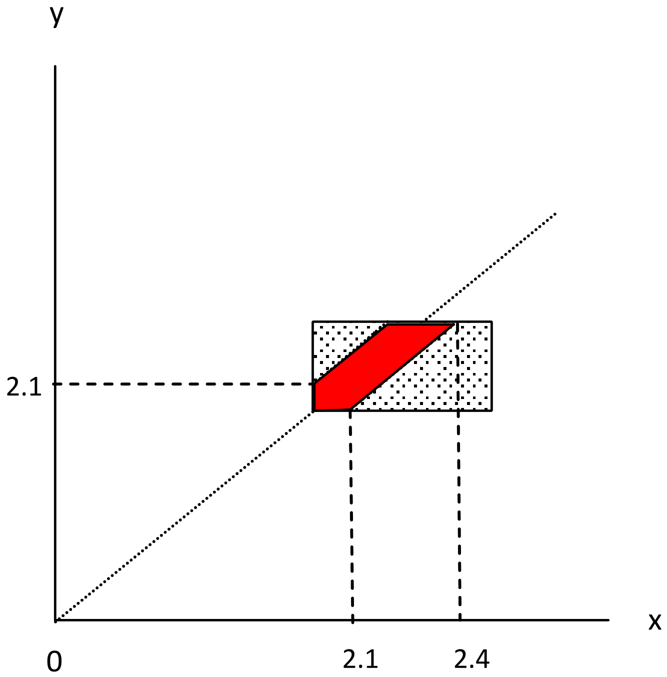
$$
\begin{aligned}
f_y(y) &= \int_2^{2.5} \frac{20}{3} dx \\
f_y(y) &= 3\frac{1}{3} \text{ for } 2 \leq y \leq 2.3
\end{aligned}
$$

(c) Write out the conditional pdf for X given $Y = y$.

$$
g(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{\frac{20}{3}}{\frac{10}{3}} = 2 \text{ for } 2 \leq x \leq 2.5
$$

we see $g(x|y) = f_x(x)$. X and Y are independent.

(d) For any X, the value of Y must be $X \pm .1$. Likewise for any Y.



(e) The probability that a rebid is required can be determined by calculating the area associated with $|X - Y| \leq .1$ (shown above) relative to the entire area of the sample space. This will work because $f(x, y)$ is constant for all (x, y).

The area associated with $2 \leq x \leq 2.5$ and $2 \leq y \leq 2.3$ is $(0.5)(0.3) = .15$

The area not associated with $|X - Y| \leq .1$ is equal to the upper triangle: $\frac{1}{2}(.2)(.2) = 0.02$ and the lower region: $\frac{1}{2}(.4 + .1)(.3) = 0.075$

$\mathbb{P}\left(\text{rebid}\right) = 1 - \frac{.02+.075}{.15} = 0.366\,67$

12. Prove the following: If $X \sim N(\mu, \sigma^2)$ then if $Z = \frac{X-\mu}{\sigma}$, Z will be distributed $Z \sim N(0,1)$.
Proof: $X \sim N(\mu, \sigma^2)$ so X is a random variable with known density $f(x)$ (and CDF $F(x)$). $Z = u(X) = \frac{X-\mu}{\sigma} = -\frac{\mu}{\sigma} + \frac{1}{\sigma}X$ is a linear function of X. $u(X)$ is invertible with inverse $X = v(Z) = \mu + \sigma Z$ and $v'(Z) = \sigma$.
The density of X is $f(x)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Using the change of variable technique we can write the pdf of $Z, \phi(Z)$ as

$$\phi(z) = f(v(z))v'(z)$$

where

$$f(v(z)) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-((\mu+\sigma Z)-\mu)^2}{2\sigma^2}\right)$$

and

$$v'(z) = \sigma$$

so

$$\begin{aligned} \phi(z) &= \left(\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-((\mu+\sigma Z)-\mu)^2}{2\sigma^2}\right)\right)\sigma \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(\sigma Z)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\sigma^2(Z)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-Z^2}{2}\right) \end{aligned}$$

this is the density of a normally distributed random variable with mean 0 and variance 1.

13. Find the density function $g(y)$ of the following functions of a single random variable:

(a) $Y = 10 - 3X$ when $X \sim N(-10, 25)$

Y is a linear function of a normally distributed random variable so Y will be normally distributed with mean

$$\mu_Y = 10 - 3\mu_X = 10 - 3(-10) = 40$$

and variance

$$\sigma_Y^2 = (-3)^2\sigma_X^2 = 9(25) = 225$$
$$Y \sim N(40, 225)$$

or

$$g(y) = \frac{1}{15\sqrt{2\pi}} \exp\left(-\frac{(y-40)^2}{450}\right) \qquad -\infty < y < \infty$$

(b) $Y = 4X$ when $X \sim b(25, .25)$

X is distributed as binomial with

$$f(x) = \binom{25}{x} .25^x .75^{25-x}$$

and

$$F(x) = \sum_{t=0}^{x} \binom{25}{t} .25^t .75^{25-t}$$

with

$$\begin{aligned} Y &= u(X) = 4X \\ \text{and } X &= 0, 1, 2, 3, \ldots, 25 \end{aligned}$$

then the random variable Y will have support

$$Y = 0, 4, 8, 12, \ldots, 100$$

$$X = v(Y) = \frac{1}{4}Y$$

$$\begin{aligned} G(Y) &= \mathbb{P}\left(Y \leq y\right) \\ &= \mathbb{P}\left(4X \leq y\right) \\ &= \mathbb{P}\left(X \leq \frac{1}{4}y\right) \\ &= F(\frac{y}{4}) \end{aligned}$$

with

$$\begin{aligned} G(y) &= F\left(\frac{Y}{4}\right) \\ g(y) &= f(\frac{Y}{4}) \\ &= \binom{25}{y/4} (.25)^{\frac{y}{4}} (.75)^{25-\frac{y}{4}} \end{aligned}$$

(c) $Y = e^X$ when $X \sim N(\mu, \sigma^2)$

X is $N(\mu, \sigma^2)$ so the pdf can be written

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right)$$

with $Y = e^X$, this function is monotonic (increasing) and the inverse can be written

$$X = v(Y) = \ln Y$$

and

$$v'(Y) = \frac{1}{Y}$$

using the change of variable technique

$$\begin{aligned} g(y) &= f(v(y))|v'(y)| \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma}\right)\frac{1}{y} \\ &= \frac{1}{\sigma y\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma}\right) \end{aligned}$$

(d)

$$Y = \begin{cases} 1 & X \geq 7.4 \\ 0 & X < 7.4 \end{cases} \quad \text{when } X \sim N(13, 20)$$

Given the function, Y will be a discrete random variable taking a value of 1 when X equals or exceeds 7.4 and taking a value of 0 when X is less than 7.4. The probability that $Y = 1$ can be found from

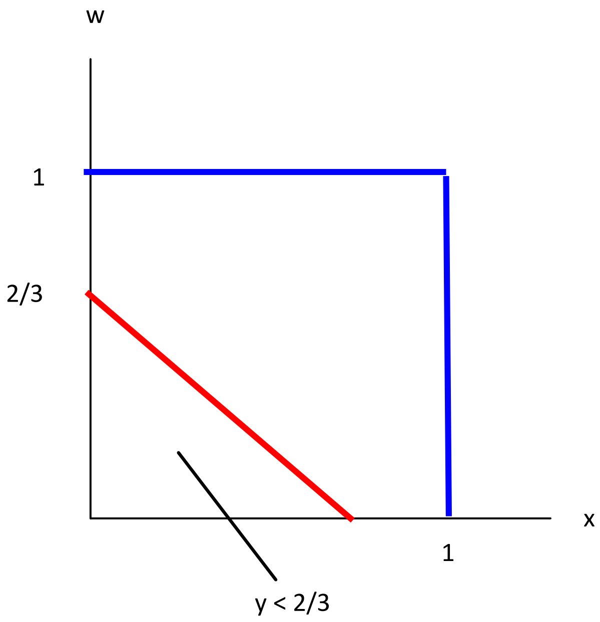
$$\begin{aligned} \mathbb{P}(Y = 1) &= \mathbb{P}(X \geq 7.4) \\ &= \mathbb{P}\left(\frac{X - 13}{\sqrt{20}} \geq \frac{7.4 - 13}{\sqrt{20}}\right) \\ &= \mathbb{P}(Z \geq -1.25) \\ &= 1 - \mathbb{P}(Z < -1.25) \\ &= 1 - \Phi(-1.25) \\ &= \Phi(1.25) \\ &= .8944 \end{aligned}$$

so we have

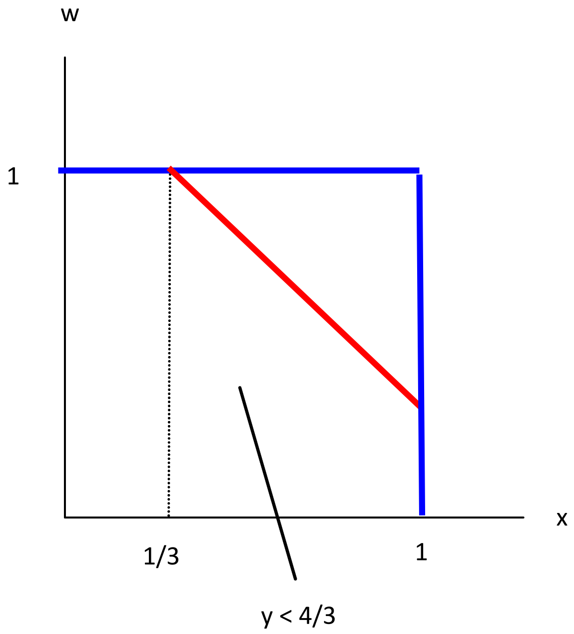
$$g(y) = \begin{cases} .8944 & y = 1 \\ .1056 & y = 0 \end{cases}$$

14. Let X and W be uniformly distributed random variables, $X \sim U(0, 1)$ and $W \sim U(0, 1)$. Let Y be a function of these random variables with $Y = X + W$.

(a) Show a sketch of the sample space and show the probability associated with $Y \leq \frac{2}{3}$.

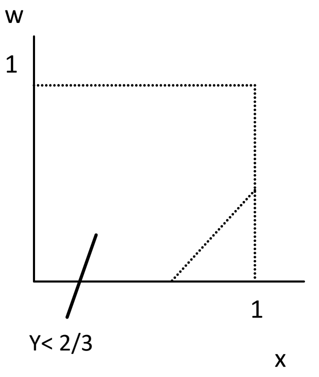


(b) Show a sketch of the sample space and show the probability associated with $Y \leq \frac{4}{3}$.



15. Let X and W be independent, uniformly distributed random variables, $X \sim U(0,1)$ and $W \sim U(0,1)$. Let Y be a function of these random variables with $Y = X - W$.

(a) For $Y \leq \frac{2}{3}$: $X - W \leq \frac{2}{3}$, all $W \geq X - \frac{2}{3}$



Note that the distribution of Y can be deerived based on a consideration of areas in the above figure.

For $0 \leq y \leq 1$ we can derive the CDF, $G(y)$ as the total area (1) minus the small triangle in the bottom right corner: $G(y) = 1 - \frac{1}{2}(1-y)(1-y) = 1 - \frac{1}{2}(y-1)^2$

For $-1 \leq y \leq 0$, the cumulative $G(y)$ will now be the area of the upper-left triangle: $G(y) = \frac{1}{2}(1+y)(1+y) = \frac{1}{2}(y+1)^2$

$$G(y) = \begin{cases} 1 - \frac{1}{2}(y-1)^2 & 0 \leq y \leq 1 \\ \frac{1}{2}(y+1)^2 & -1 \leq y \leq 0 \end{cases}$$

The density function will be $g(y) = G'(y)$

$$g(y) = \begin{cases} 1 - y & 0 \leq y \leq 1 \\ y + 1 & -1 \leq y \leq 0 \end{cases}$$