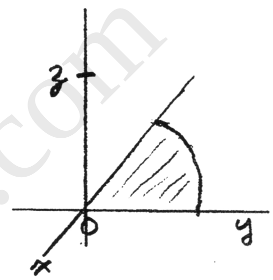
75 minutes. Closed book. Show your work in the space provided. All questions carry equal weight. See formula sheets pp. 7-8. Non-graphical/non-programmable calculators only.

1. A region in the x-y plane in the shape of a quarter-disc is defined by $0 < \rho < b$, $\pi/2 < \phi < \pi$, and $z = 0$, where (ρ, ϕ, z) are the usual cylindrical coordinates. In this region the surface charge density ρ_s depends on azimuthal angle ϕ as $\rho_s = c\phi$, where c is a constant and ϕ is in radians. The charge density outside of this region is zero. Find the z-component of the electric field at a point z on the z-axis.



$$E_z = \vec{E}\cdot\hat{a}_z = \frac{1}{4\pi\varepsilon_0}\int ds \frac{\rho_s}{|\vec{r}-\vec{r}'|^2}\hat{a}_{rr'}\cdot\hat{a}_z \quad \text{where } ds = d\rho\cdot\rho d\varphi$$

and $\vec{r} = z\hat{a}_z$, $\vec{r}' = \rho\hat{a}_\rho$, $\hat{a}_{rr'} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} = \frac{z\hat{a}_z - \rho\hat{a}_\rho}{(z^2+\rho^2)^{1/2}}$

$$\therefore E_z = \frac{1}{4\pi\varepsilon_0}\int_0^b d\rho\,\rho\int_{\pi/2}^{\pi} d\varphi\, \frac{c\varphi}{z^2+\rho^2}\,\frac{z}{(z^2+\rho^2)^{1/2}} \leftarrow \hat{a}_{rr'}\cdot\hat{a}_z = z(z^2+\rho^2)^{-1/2}$$

$$= \frac{cz}{4\pi\varepsilon_0}\int_{\pi/2}^{\pi} d\varphi\cdot\varphi\int_0^b d\rho\,\rho(z^2+\rho^2)^{-3/2}$$

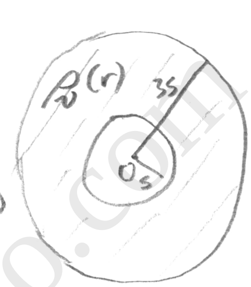
$$= \frac{cz}{4\pi\varepsilon_0}\left[\frac{1}{2}\varphi^2\right]_{\pi/2}^{\pi}\cdot\left[-(z^2+\rho^2)^{-1/2}\right]_0^b$$

$$= \frac{cz}{4\pi\varepsilon_0}\frac{1}{2}\underbrace{\left(\pi^2-\frac{\pi^2}{4}\right)}_{\frac{3}{4}\pi^2}\left(-(z^2+b^2)^{-1/2}+\underbrace{(z^2)^{-1/2}}_{|z|^{-1}}\right)$$

$$= \frac{3\pi c}{32\varepsilon_0}z\left(\frac{1}{|z|}-\frac{1}{(z^2+b^2)^{1/2}}\right)$$

2. An uncharged, spherical conductor of radius s is centred on the origin. Outside the conductor, the region of space $s < r < 3s$ is filled with a charge distribution with density $\rho_v = a/r^2$, where a is a constant and r is the distance from the origin. The region $r > 3s$ has $\rho_v = 0$.
   a) Use Gauss' law to find the electric field $\vec{E}$ everywhere in space, that is, for (i) $r < s$, (ii) $s < r < 3s$, and (iii) $3s < r$. (Answer in terms of variables given above, showing your work.)
   b) If the radius of the conductor is $s = 2$ cm, and the magnitude of the electric field at a radius of 5 cm from the centre of the sphere is 8 V/m, find the value of the constant a.

a) $\vec{E}(\vec{r}) = E(r)\,\hat{a}_r$ by symmetry



$\therefore$ Gauss' law $\oint_S \vec{E}\cdot d\vec{s} = \dfrac{Q_{enc}}{\varepsilon_0}$ for S = sphere of radius r about 0 with $d\vec{s} = \hat{a}_r ds$

$$\Rightarrow E(r)\underbrace{\oint_S ds}_{4\pi r^2} = Q_{enc}/\varepsilon_0 \quad \Rightarrow E(r) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}$$

(i) $r<s$: $Q_{enc} = 0 \Rightarrow E(r) = 0 \Rightarrow \vec{E} = 0$

(ii) $s<r<3s$: $Q_{enc} = \displaystyle\int_{r'<r} dv'\,\rho_v(r') = \int_s^r 4\pi r'^2 dr'\,(a/r'^2)$

$$= 4\pi a r'\Big|_s^r = 4\pi a(r-s)$$

$$\therefore E(r) = \frac{4\pi a(r-s)}{4\pi\varepsilon_0 r^2} = \frac{a(r-s)}{\varepsilon_0 r^2}$$

(set $r=3s$ above)

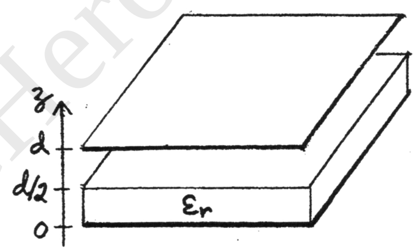
(iii) $r>3s$: $Q_{enc} = \displaystyle\int_{r'<3s} dv'\,\rho_v(r') = 4\pi a(3s-s) = 8\pi a s$

$$\therefore E(r) = 8\pi a s/(4\pi\varepsilon_0 r^2) = \frac{2as}{\varepsilon_0 r^2}$$

b) $s = 2\text{cm}$, $E = 8\,\text{V/m}$ at $r = 5\text{cm}$, so $s<r<3s$, so (ii) gives

$$a = \frac{\varepsilon_0 r^2 E(r)}{r-s} = \frac{(8.85\cdot 10^{-12}\,\text{F/m})(.05\,\text{m})^2(8\,\text{V/m})}{(.05-.02)\,\text{m}} = 5.9\cdot 10^{-12}\ \text{C/m}$$

3. A parallel plate capacitor is made up of two flat conducting plates, each of area A, separated by a distance d. The lower plate is in the $z = 0$ plane and carries charge $-q$. The upper plate is in the $z = d$ plane and carries charge q. In between the plates, an insulating material with dielectric constant $\epsilon_r = 5$ fills the region from $z = 0$ to $z = d/2$, while the region from $z = d/2$ to $z = d$ is vacuum. (Solve this problem from basic principles given on the formula sheet, without using the series capacitor circuit formula. Assume the area A of the plates is large, so you may ignore any deviation of the fields near the edges of the plates.)
   a) Find $\vec{D}$ everywhere between the plates.
   b) Find $\vec{E}$ and $\vec{P}$ in the insulating dielectric material.
   c) Find the bound surface charge density ρ_{sb} at (i) the top and (ii) the bottom of the dielectric material.
   d) Find the energy stored in the capacitor (equivalent to the work of formation of the free charge distribution on the plates) by integrating the appropriate energy density over the volume between the plates of the capacitor. (Express your answer in terms of the variables given in the description of the problem.)



a) $D_n = \rho_s$ at surface of conductor, and
e.g. at bottom of upper plate $\hat{a}_n = -\hat{a}_z$
so with $\rho_s = q/A$

$\therefore D_n = \rho_s = q/A \Rightarrow D_z = -D_n = -q/A$ and $\vec{D} = D_z \hat{a}_z$

b) $\vec{D} = \varepsilon\vec{E} \Rightarrow \vec{E} = \frac{1}{\varepsilon}\vec{D} = \frac{1}{5\varepsilon_0}\left(-\frac{q}{A}\hat{a}_z\right) = -\frac{q}{5\varepsilon_0 A}\hat{a}_z$

$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \vec{P} = \vec{D} - \varepsilon_0\vec{E} = \left(-\frac{q}{A} - \left(-\frac{q}{5A}\right)\right)\hat{a}_z = -\frac{4}{5}\frac{q}{A}\hat{a}_z$

c) (i) $\rho_{sb} = \vec{P}\cdot\hat{a}_n = -\frac{4}{5}\frac{q}{A}\hat{a}_z\cdot\hat{a}_z = -\frac{4}{5}\frac{q}{A}$

(ii) $\rho_{sb} = \vec{P}\cdot\hat{a}_n = -\frac{4}{5}\frac{q}{A}\hat{a}_z\cdot(-\hat{a}_z) = \frac{4}{5}\frac{q}{A}$

d) $W = \frac{1}{2}\int dv\, \vec{D}\cdot\vec{E} = \frac{1}{2}\int dv\, \vec{D}\cdot\left(\frac{\vec{D}}{\varepsilon}\right) = \frac{1}{2}\int_{0<z<d/2} dv\, \frac{D^2}{5\varepsilon_0} + \frac{1}{2}\int_{\frac{d}{2}<z<d} dv\, \frac{D^2}{\varepsilon_0}$

$= \frac{1}{2}\left(\frac{d}{2}A\right)\left(\frac{D^2}{5\varepsilon_0}\right) + \frac{1}{2}\left(\frac{d}{2}A\right)\left(\frac{D^2}{\varepsilon_0}\right)$ where $D = q/A$

$= \frac{1}{2}\frac{d}{2}A\left(\frac{q}{A}\right)^2\frac{1}{\varepsilon_0}\underbrace{\left(\frac{1}{5}+1\right)}_{\frac{6}{5}} = \frac{3dq^2}{10\varepsilon_0 A}$