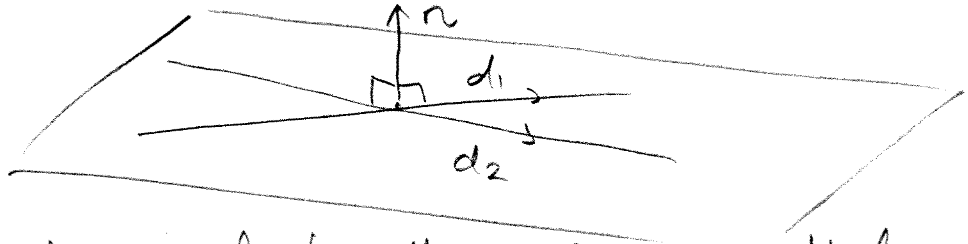
1241A F17 V.1 soln

1. An equation for the plane which contains the two lines with parametric equations $x = -1 + t,\ y = -1 - t,\ z = 1 + 3t$ and $x = -3 - s,\ y = 3 + 2s,\ z = 7 + 3s$, is:

A. $7x - 11y + 2z = 6$
B. $11x - 2y + 9z = 0$
C. $6x - 2y + z = -3$
D. $3x - 6y + z = 4$
E. $9x + 6y - z = -16$
F. $9x + 6y + z = -14$



A normal for this plane will be $\pm\, d_1 \times d_2$, where $d_1 = (1,-1,3)$ and $d_2 = (-1,2,3)$ are direction vectors for the lines above. A computation shows

$$d_1 \times d_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -1 & 2 & 3 \end{vmatrix} = (-9,-6,1).$$

So we take $n = (9,6,-1)$. Since the plane contains $(-1,-1,1)$, $n \cdot (-1,-1,1) = -16$. Hence an equation for the plane is

$$9x + 6y - z = -16$$

2. An equation for the plane passing through the points $\overset{P}{(0,-3,0)}$ and $\overset{Q}{(-1,1,2)}$, and which is parallel to the x–axis is:

A. $3x + 2y + 7z = -6$
B. $2x - y = 3$
C. $x - y + z = 3$
D. $x - z = 0$
E. $y - 2z = -3$
F. $x + y + z = -3$

Such a plane has a normal which is $\perp$ to both $P - Q$ and $(1,0,0)$, and so the normal would be parallel to their cross product.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -2 \\ 1 & 0 & 0 \end{vmatrix} = (0,-2,4).$$

So we take $n = (0,1,-2)$ and note there is only one plane in the list with this normal: $y - 2z = -3$, (which we can check contains both points if we wish.)

3. Find an equation of the plane which passes through the point $(1, 1, 1)$ and which is perpendicular to the line whose scalar parametric equations are:

$$x = -6 + 2t,\ y = 1 - 4t,\ z = -3 + 3t;\ t \in \mathbf{R}.$$

A. $2x - 4y + 3z = 1$
B. $2x + 4y + 3z = 9$
C. $6x + y - 3z = 2$
D. $2x - 4y + 3z = -25$
E. $2x - 4y + 3z = -10$
F. $2x - 4y + 3z = 10$

A normal for this plane will be a direction vector for this line, which is $(2, -4, 3)$. Since the plane contains $(1, 1, 1)$,

$n \cdot (1, 1, 1) = (2, -4, 3) \cdot (1, 1, 1) = 1$. Hence

an equation for the plane is $2x - 4y + 3z = 1$.

4. Parametric equations for the line containing $\overset{P}{(2,\ -2,\ 3)}$ and $\overset{Q}{(-2,\ 4,\ 0)}$ are:

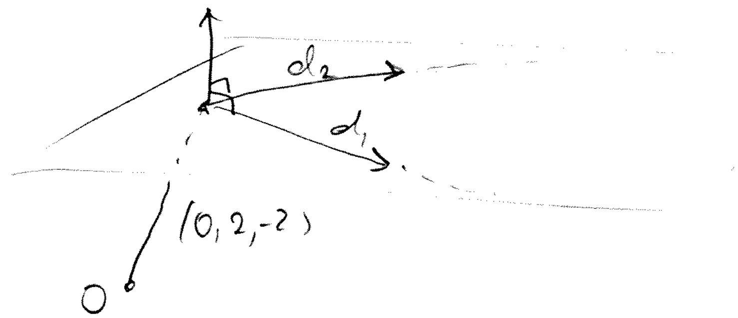
A. Such a line does not exist.
B. $x = 2 - 2t,\ y = -2 + 4t,\ z = 3;\ t \in \mathbf{R}$.
C. $x = 1 - t,\ y = -1 - 6t,\ z = 4 + 3t;\ t \in \mathbf{R}$.
D. $x = 3 + 4t,\ y = -1 - 6t,\ z = 6 + t;\ t \in \mathbf{R}$.
E. $x = 2 + 4t,\ y = -2 - 6t,\ z = 3 + 3t;\ t \in \mathbf{R}$.
F. $x = -2 + 4t,\ y = 4 + 6t,\ z = 1 + 3t;\ t \in \mathbf{R}$.

A direction vector for such a line is $P - Q = (4, -6, 3)$.

There is only one line above with this direction vector, namely "E", which does indeed (one can check) pass through both P & Q.

5. Find a Cartesian (scalar) equation for the plane with vector parametric equation

$$v = (0, 2, -2) + s\underbrace{(1, -1, 2)}^{d_1} + t\overbrace{(2, -4, -1)}^{d_2};\ s, t \in \mathbf{R}.$$



A. $4x - 9y + 6z = -30$

B. $9x + 5y - 2z = 14$

C. $9x - 5y + 2z = -14$

D. $9x + 5y + 2z = 6$

E. $9x + 2y + 5z = -6$

F. $9x - 2y + 5z = -14$

A normal for such a plane will be parallel to $d_1 \times d_2$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & -4 & -1 \end{vmatrix} = (9, +5, -2).$$

There is only one plane in the list above with this normal, namely that with eqn $9x + 5y - 2z = 14$. One can check that this does indeed contain the point $(0,2,-2)$.

6. Find all vectors in $\mathbf{R}^3$ which are perpendicular to both $\overset{u}{(-1, 1, 5)}$ and $\overset{v}{(2, 1, 2)}$.

A. $\{(2,\ -8,\ 2)\}$

B. $\{(t+1,\ -8,\ t+1) \mid t \in \mathbf{R}\}$

C. $\{(t,\ -4t,\ t) \mid t \in \mathbf{R}\}$

D. $\{(-t,\ 0,\ t) \mid t \in \mathbf{R}\}$

E. $\{(0,\ 0,\ 0)\}$

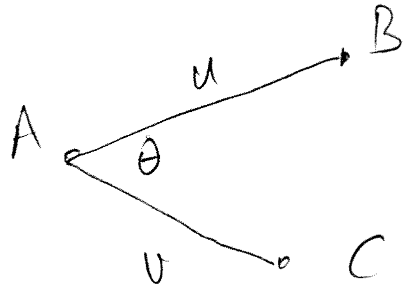
F. $\{(3,\ -12,\ 3)\}$

Such vectors will be parallel to $u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 5 \\ 2 & 1 & 2 \end{vmatrix} = (-3, +12, -3)$

Hence C. is correct.

7. A triangle has vertices $A = (1,\ 1,\ 1)$, $B = (2,\ 3,\ 1)$ and $C = (1,\ 2,\ 3)$. Find the cosine of the interior angle at A.

A. 0
B. 1/5
C. 2/5
D. 3/5
E. 4/5
F. 1



$u = B - A = (1, 2, 0)$

$v = C - A = (0, 1, 2)$

Hence $\cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|} = \dfrac{2}{\sqrt{5} \cdot \sqrt{5}} = \dfrac{2}{5}$

8. The point of intersection of the line through the point $(2,\ 1,\ 0)$ parallel to the vector $u = (1,\ -1,\ 2)$ with the plane with equation $x + y + 2z = 23$ is:

A. (11, 4, 4)
B. (2, 1, 10)
C. (7, −4, 10)
D. (7, 4, 6)
E. (11, 1, 4)
F. (10, −4, 7)

The line will have scalar parametric form $x = 2 + s,\ y = 1 - s,\ z = 2s;\ s \in \mathbb{R}$.

Since in addition $x + y + 2z = 23$, we find $(2+s) + (1-s) + 2(2s) = 23$, yielding $4s = 20$ or $s = 5$. Hence the point is $(7, -4, 10)$.

9. If $u = (3,\ 3,\ 6)$ and $v = (2,\ -1,\ 1)$ then the length of the projection of u along v is:

A. $\dfrac{3\sqrt{6}}{2}$

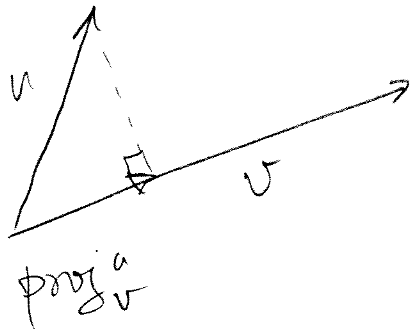
B. $\dfrac{3\sqrt{2}}{2}$

C. 0

D. $\dfrac{\sqrt{6}}{2}$

E. $\dfrac{2\sqrt{6}}{3}$

F. $\dfrac{2\sqrt{2}}{3}$



Since $\text{proj}_v u = \dfrac{u\cdot v}{\|v\|^2}\, v$

$$\|\text{proj}_v u\| = \frac{|u\cdot v|\,\|v\|}{\|v\|^2} = \frac{|u\cdot v|}{\|v\|} = \frac{9}{\sqrt{6}} = \frac{\sqrt{6}\cdot 9^{3}}{6_{2}} = \frac{3\sqrt{6}}{2}$$

10. Find the area of the triangle whose vertices are the points $P = (3,\ -1,\ 2)$, $Q = (1,\ 1,\ 0)$ and $R = (1,\ 2,\ -1)$.

A. 4

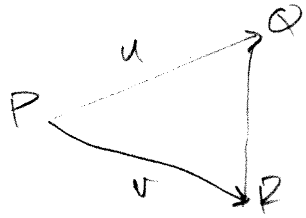
B. $2\sqrt{2}$

C. $\sqrt{2}$

D. 0

E. $4\sqrt{2}$

F. 2



The area is $\frac{1}{2}\|u\times v\|$.

Since $u = Q-P = (-2,2,-2)$ and $v = R-P = (-2,3,-3)$, we find

$$u\times v = 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -1 \\ -2 & 3 & -3 \end{vmatrix} = 2(0,-1,-1) \quad \therefore \frac{1}{2}\|u\times v\| = \sqrt{2}$$

11. Write the complex number

$$\frac{(1+3i)\,(5+10i)}{4+3i} = \frac{(-25+25i)(4-3i)}{25}$$

in Cartesian form: $a+bi$, with $a, b \in \mathbf{R}$.

$= (-1+i)(4-3i)$

$= -1+7i$

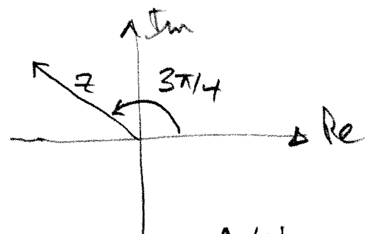
A. $-1+7i$
B. -1
C. $1+7i$
D. $7i$
E. $-5+35i$
F. $-\frac{1}{5}+\frac{7}{5}i$

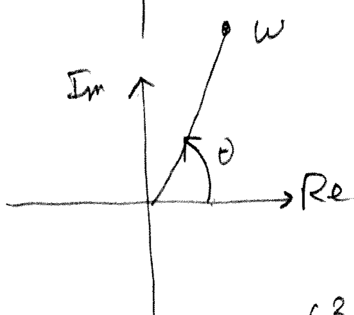
12. What is the polar form of $\dfrac{-\sqrt{2}+\sqrt{2}i}{3+3\sqrt{3}i}$?

If $z = -\sqrt{2}+\sqrt{2}i$ and $w = 3+3\sqrt{3}i$

then $|z| = 2$, and $z = 2e^{\frac{3\pi i}{4}}$.

A. $\frac{1}{3}(\cos(5\pi/12) + i\sin(5\pi/12))$
B. $\frac{1}{3}(\cos(5\pi/12) - i\sin(5\pi/12))$
C. $3(\cos(5\pi/12) - i\sin(5\pi/12))$
D. $3(\cos(5\pi/12) + i\sin(5\pi/12))$
E. $\cos(11\pi/12) + i\sin(11\pi/12)$
F. $\cos(5\pi/12) + i\sin(5\pi/12)$





Moreover,

$|w| = \sqrt{9+27} = 6$

and $\cos\theta = \frac{3}{6} = \frac{1}{2}$

$\sin\theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$

Hence $\theta = \pi/3$. Thus $\frac{z}{w} = \frac{2}{6} e^{\pi i\left(\frac{3}{4}-\frac{1}{3}\right)} = \frac{1}{3} e^{\frac{5\pi i}{12}}$