Student No.:____________________________

Name: ___________________________________

FACULTY OF ENGINEERING AND COMPUTER SCIENCE
CONCORDIA UNIVERSITY

MIDTERM EXAMINATION

COURSE: ENGR 311 — TRANSFORM CALCULUS
INSTRUCTOR: M. ESHAGHI — MAX. MARKS: 100
DATE: 27 NOVEMBER 2018 — DURATION: 75 MINUTES

- *Write directly on this questions booklet. Show your work in sufficient details and clearly to qualify for partial marks*
- *DO NOT UNSTAPLE THE QUESTIONS BOOKLET*
- *YOU CAN FIND FORMULA SHEET IN THE LAST PAGE*
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Question #1 (15 Marks)

1.1. A certain function, $g(x)$, has Fourier Series as follows,

$$f(x) = 1 + \frac{\cos(3\pi x)}{4} + \frac{\cos(6\pi x)}{8} + \frac{\cos(9\pi x)}{12} + \cdots$$

a- Find fundamental period of the Fourier series

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{4n} \cos(3n\pi x) = 1 + \sum_{n=1}^{\infty} \frac{1}{4n} \cos \frac{n\pi}{1/3} x$$

4

$$P = \frac{1}{3} \rightarrow T = \frac{2}{3}$$

b- Find Fourier Series of $4g(x) + 101$

$$4g(x) + 101 = 101 + 4\left(1 + \frac{\cos 3\pi x}{4} + \frac{\cos 6\pi x}{8} + \cdots\right)$$

4

$$= 105 + \cos 3\pi x + \frac{\cos 6\pi x}{2} + \cdots$$

1.2. Fourier Series of following function is in the following form,

$$f(t) = A + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{p} t\right)$$

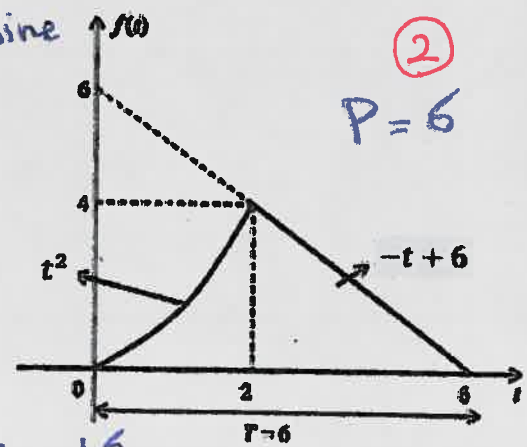
Find coefficient A.

The given series is Fourier cosine

2

$$f(t) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi}{p} t$$

$P = 6$



$$A = \frac{a_0}{2}$$

5

$$a_0 = \frac{2}{p} \int_0^p f(t)\,dt = \frac{2}{6}\left(\int_0^2 t^2\,dt + \int_2^6 (-t+6)\,dt\right)$$

$a_0 = 32/9$ $\qquad$ $A = 16/9$

Question #2 (25 Marks)

Show that the given set of functions is orthogonal on the indicated interval. Then, construct an orthonormal set from the given set.

$$\{5, \cos(n\pi x)\}, \quad n = 1,2,3, \ldots \quad [0,1]$$

$$(5, \cos(n\pi x)) = \int_0^1 5 \cos n\pi x \, dx = 0$$ 5

$$(\cos n\pi x, \cos m\pi x) = \int_0^1 \cos n\pi x \cos m\pi x \, dx =$$

$$\frac{1}{2}\int_0^1 \cos \pi(n+m)x \, dx + \frac{1}{2}\int_0^1 \cos \pi(n-m)x \, dx$$ 5

$$= \left[\frac{1}{2\pi(n+m)} \sin \pi(n+m)x\right]_0^1 + \left[\frac{1}{2\pi(n-m)} \sin \pi(n-m)x\right]_0^1$$

$$0 \qquad\qquad 0$$

$$= 0$$

As a result the set is orthogonal

$$(5,5) = \int_0^1 25 \, dx = 25 \rightarrow \text{Norm} = \sqrt{25} = 5$$ 5

$$(\cos n\pi x, \cos n\pi x) = \int_0^1 \cos^2 n\pi x \, dx = \frac{1}{2}\int_0^1 (1 + \cos 2n\pi x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2n\pi} \sin 2n\pi x\right)_0^1 = \frac{1}{2} \rightarrow \text{Norm} = \sqrt{1/2}$$ 5

Normalized set $= \left\{ \frac{5}{5}, \frac{\cos n\pi x}{\sqrt{1/2}} \right\}$ 5

Question #3 (25 Marks)

Consider the following function

$$f(x) = \begin{cases} 1 & 0 < x \leq 2 \\ -3 & 2 < x \leq 3 \end{cases}$$

a) Find the Fourier Cosine Series of $f(x)$ and sketch it on the interval [−9,9]

b) To what values will this series converge at $x = 0$, $x = 7$ and $x = -9$?

a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{P} x$ 1 $P = 3$ 2

$a_0 = \frac{2}{P}\int_0^P f(x)\,dx = \frac{2}{3}\left(\int_0^2 dx + \int_2^{+3} -3\,dx\right) = -\frac{2}{3}$ 4

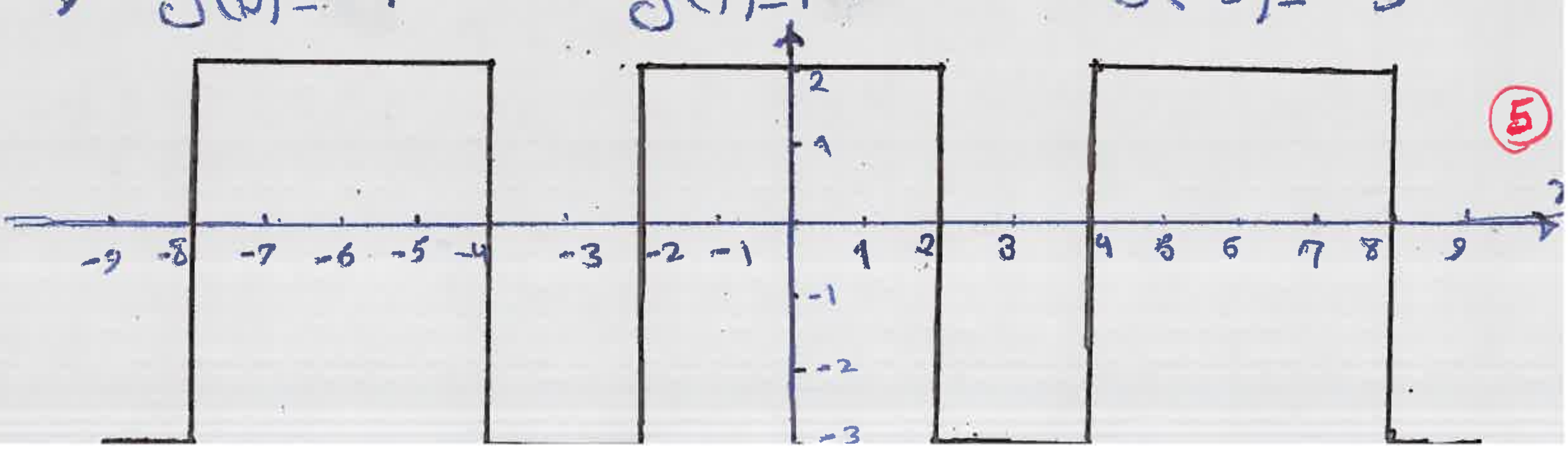
$a_n = \frac{2}{P}\int_0^P f(x) \cos \frac{n\pi}{P} x\,dx = \frac{2}{3}\int_0^2 \cos \frac{n\pi}{3} x\,dx +$

$\frac{2}{3}\int_2^3 -3 \cos \frac{n\pi}{3} x\,dx = \frac{2}{n\pi}\left(\sin \frac{2n\pi}{3}\right) + \frac{6}{n\pi}\left(\sin \frac{2n\pi}{3}\right.$ 7

$a_n = \frac{8}{n\pi} \sin \frac{2n\pi}{3}$

$f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty}\left(\frac{8}{n\pi} \sin \frac{2n\pi}{3} \cos \frac{n\pi}{3} x\right)$

b) 2 $g(0) = 1$ 2 $g(7) = 1$ 2 $g(-9) = -3$



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Question #4 (35 Marks)

Solve following wave equation,

$$4\,u_{xx} = u_{tt} \qquad 0 < x < 1$$

$$B.C. \qquad u(0,t) = 0, \qquad u(1,t) = 0$$

$$I.C. \qquad u(x,0) = \sin(\pi x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin(2\pi x)$$

Find displacement of the string in $x = 0.25$ m and $t = 0.2$ s.
Note: λ cannot be zero or negative. As a result, solve the equation assuming λ is positive.

$u(x,t) = X(x)\,T(t)$ 1 $\qquad u_{xx} = X''T \qquad u_{tt} = XT''$

$4X''T = XT'' \Rightarrow \dfrac{X''}{X} = \dfrac{T''}{4T} = -\lambda \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T'' + 4\lambda T = 0 \end{cases}$ 3

$u(0,t) = 0 \rightarrow X(0) = 0$ 2 $\qquad u(1,t) = 0 \rightarrow X(1) = 0$ 2

$\lambda = \alpha^2 \qquad X'' + \alpha^2 X = 0 \rightarrow X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$ 2

$X(0) = 0 \rightarrow c_1 = 0$ 2

$X(1) = 0 \rightarrow c_2 \sin\alpha = 0 \rightarrow \alpha = n\pi$ 2

$X(x) = c_2 \sin n\pi x$ 1

$T'' + 4\lambda T = 0 \quad$ or $\quad T'' + 4n^2\pi^2 T = 0$

$T(t) = c_3 \cos 2n\pi t + c_4 \sin 2n\pi t$ 2

$u_n(x,t) = c_2 \sin n\pi x\,(c_3 \cos 2n\pi t + c_4 \sin 2n\pi t)$ 2

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos 2n\pi t + B_n \sin 2n\pi t\right) \sin n\pi x$$ 2

$$u(x,0) = \sum A_n \sin n\pi x = \sin \pi x$$ 2

$A_1 = 1$ 2 $\quad A_n = 0$ 1 $\quad n \neq 1$

2

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n (2n\pi) \sin 2n\pi t + B_n (2n\pi) \cos 2n\pi t\right) \sin$$

2

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n (2n\pi) \sin n\pi x = \sin 2\pi x$$

$\Longrightarrow B_2 (4\pi) = 1 \qquad B_n = 0 \quad n \neq 2$

$B_2 = \frac{1}{4\pi}$ 2

$$u(x,t) = \cos 2\pi t \sin \pi x + \frac{1}{4\pi} \sin 4\pi t \sin 2\pi x$$

2

$u(0.25, 0.2) = 0.265$ 1