Question #1 (25 Points)

In the following question,

a) Solve following initial value problem.

$$y'' - 4y' + 5y = g(t), \qquad y(0) = 0, \qquad y'(0) = 1, \qquad g(t) = \begin{cases} 5 & t < 1 \\ 0 & t \geq 1 \end{cases}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \quad 1$$

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^1 5e^{-st}\,dt + \cancel{\int_1^\infty 0 e^{-st}\,dt} = -\frac{5}{s}(e^{-s} - 1) \quad 2$$

$$s^2 Y - s\cancel{y(0)} - y'(0) - 4sY - 4\cancel{y(0)} + 5Y = \frac{5}{s} - e^{-s}\frac{5}{s} \quad 1$$

$$(s^2 - 4s + 5)Y - 1 = \frac{5}{s} - e^{-s}\frac{5}{s} \Rightarrow Y(s) = \frac{1 + \frac{5}{s} - e^{-s}\frac{5}{s}}{s^2 - 4s + 5}$$

$$Y(s) = \frac{1}{(s-2)^2 + 1} + \frac{5}{s((s-2)^2+1)} - e^{-s}\frac{5}{s((s-2)^2+1)} \quad 1.5$$

Partial Fraction:

$$\frac{5}{s((s-2)^2+1)} = \frac{A}{s} + \frac{Bs + C}{(s-2)^2+1} = \frac{A(s^2 - 4s + 5) + (Bs + C)s}{s((s-2)^2+1)} \quad 1.5$$

$$A = 1 \qquad B = -1 \qquad C = 4$$

$$Y(s) = \frac{1}{(s-2)^2+1} + \frac{1}{s} - \frac{s-2}{(s-2)^2+1} + \frac{2}{(s-2)^2+1} - e^{-s}\left(\frac{1}{s} - \frac{s-2}{(s-2)^2+1} + \frac{2}{(s-2)^2+1}\right) \quad 1$$

$$\Rightarrow y(t) = e^{2t}\sin t + 1 - e^{2t}\cos t + 2e^{2t}\sin t - u(t-1)\left(1 - e^{2(t-1)}\cos(t-1) + e^{2(t-1)}\sin(t-1)\right)$$

$$y(t) = 1 - e^{2t}\cos t + 3e^{2t}\sin t - u(t-1)\left(1 - e^{2(t-1)}\cos(t-1) + 2e^{2(t-1)}\sin(t-1)\right) \quad 4$$

b) Find Laplace transform of following function

$$f(t) = t\int_0^t \sin(2\tau)\, e^{-3\tau} d\tau$$

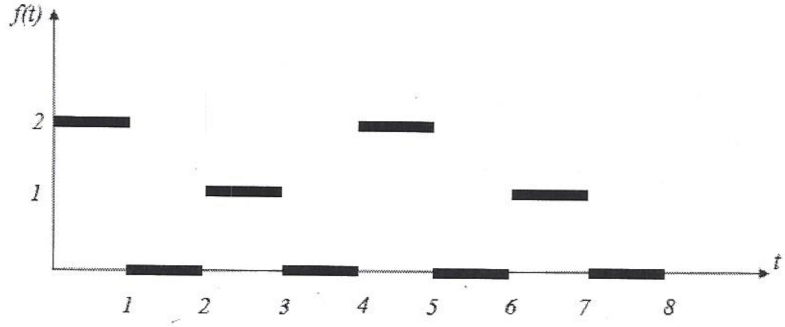
$$\mathcal{L}\{f(t)\} = -\frac{d}{ds}\mathcal{L}\left\{\int_0^t \sin(2\tau) e^{-3\tau} d\tau\right\} = \quad 1$$

$$-\frac{d}{ds}\frac{\mathcal{L}\{\sin(2t)\, e^{-3t}\}}{s} \quad 1 \quad = -\frac{d}{ds}\frac{\mathcal{L}\{\sin 2t\}_{s\to s+3}}{s}$$

$$= -\frac{d}{ds}\frac{\frac{2}{(s+3)^2+4}}{s} \quad 1 \quad = -\frac{d}{ds}\frac{2}{s((s+3)^2+4))}$$

$$\frac{2\left((s+3)^2+4+2s(s+3)\right)}{\left(s((s+3)^2+4)\right)^2} \quad 1$$

c) Find Laplace transform of following periodic function



$T = 4$ (1)

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\, dt =$$ (1)

$$\frac{1}{1-e^{-4s}} \left(\int_0^1 2e^{-st}\, dt + \int_2^3 e^{-st}\, dt \right) =$$ (1)

$$\frac{1}{1-e^{-4s}} \left(\frac{-2}{s}(e^{-s}-1) - \frac{1}{s}\left(e^{-3s}-e^{-2s}\right)\right)$$ (2)

d) Find Laplace transform of following function

$$f(t) = \int_0^t (t-\tau)^{17}\, d\tau \quad = t^{17} * 1 \quad \Rightarrow \begin{array}{l} h(t) = t^{17} \\ g(t) = 1 \end{array} \quad 1$$

$$\mathcal{L}\{f(t)\} = H(s)\ G(s)$$

$$H(s) = \frac{17!}{s^{18}} \quad 1 \qquad\qquad G(s) = \frac{1}{s} \quad 1$$

$$F(s) = \frac{17!}{s^{18}} \frac{1}{s} = \frac{17!}{s^{19}} \quad 1$$

Question #2

a) Expand the given function in a Fourier series and plot it in the range of $[-4\pi, 4\pi]$

$$f(x) = \begin{cases} 1 & 0 < x \leq \pi \\ \sin(x) & \pi < x \leq 2\pi \end{cases}$$

$L = 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \quad (1) \qquad P = \pi \quad (1)$$

$$a_0 = \frac{1}{p}\int_0^L f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \sin x\,dx = 1 - \frac{2}{\pi} \quad (2)$$

$$a_n = \frac{1}{p}\int_0^L f(x)\cos\frac{n\pi}{p}x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\sin x \cos nx\,dx$$

0

$$a_n = \frac{1}{2\pi}\int_{\pi}^{2\pi}\sin(n+1)x\,dx - \frac{1}{2\pi}\int_{\pi}^{2\pi}\sin(n-1)x\,dx$$

$$a_n = \frac{1}{2\pi}\left(\frac{-1}{n+1}\cos(n+1)x\right)_{\pi}^{2\pi} - \frac{1}{2\pi}\left(\frac{-1}{n-1}\cos(n-1)x\right)_{\pi}^{2\pi} \quad (3)$$

$$a_n = \frac{-1}{2\pi(n+1)}\left(1-(-1)^{n+1}\right) + \frac{1}{2\pi(n-1)}\left(1-(-1)^{n-1}\right) = \frac{1+(-1)^n}{\pi(n^2-1)}$$

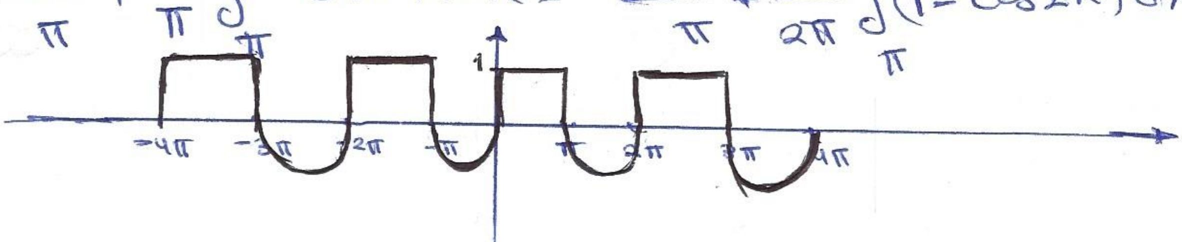
$$a_1 = \frac{1}{\pi}\int_{\pi}^{2\pi}\sin x\cos x\,dx = \frac{1}{2\pi}\int_{\pi}^{2\pi}\sin 2x\,dx = 0 \quad (2)$$

$$b_n = \frac{1}{p}\int_0^L f(x)\sin\frac{n\pi}{p}x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\sin x\sin nx\,dx \quad (3)$$

$$b_n = \frac{-1}{n\pi}(\cos n\pi - 1) + \frac{1}{2\pi}\left(\frac{1}{n+1}\sin(n+1)x\right)_{\pi}^{2\pi} + \frac{1}{2\pi}\left(\frac{1}{n-1}\sin(n-1)x\right)_{\pi}^{2\pi}$$

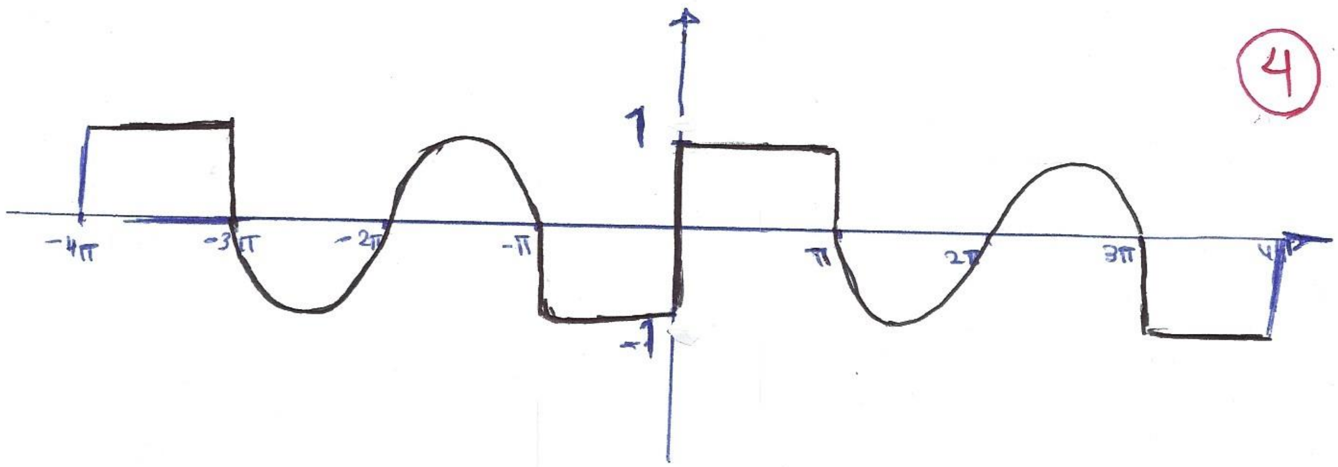
$0 \qquad 0 \qquad (2)$

$$b_1 = \frac{+2}{\pi} + \frac{1}{\pi}\int_{\pi}^{2\pi}\sin^2 x\,dx = \frac{+2}{\pi} + \frac{1}{2\pi}\int_{\pi}^{2\pi}(1-\cos 2x)\,dx = \frac{1}{2} + \frac{2}{\pi}$$



(4)

b) **WITHOUT FINDING THE SERIES** sketch the Fourier Sine Series, on a separate graph, also on the interval $[-4\pi, 4\pi]$



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c) To what values will the Fourier Sine Series converge at $x = -3\pi$, $x = 0$ and $x = 2\pi$?

$g(-3\pi) = \frac{1+0}{2} = 1/2$ 1

$g(0) = \frac{1+(-1)}{2} = 0$ 1

$g(2\pi) = 0$ 1

Question #3(25Points)

Solve following heat equation and find temperature of the rod in $x = \frac{\pi}{2}\,m$ and $t = 1\,s$.

$$\frac{\partial^2 u}{\partial x^2} + \sin(x) = \frac{\partial u}{\partial t}, \qquad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = 0 \qquad u(0,\pi) = 0$$

$$u(x,0) = 3\sin(2x)$$

$u(x,t) = v(x,t) + \psi(x)$, 1 $\qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi''(x)$, $\qquad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$

$\frac{\partial^2 v}{\partial x^2} + \psi''(x) + \sin x = \frac{\partial v}{\partial t}$ 1

$u(0,t) = v(0,t) + \psi(0) = 0 \to v(0,t)=0,\ \psi(0)=0$ 1 $\qquad v(x,0) = u(x,0) - \psi(x)$ 1

$u(\pi,t) = 0 = v(\pi,t) + \psi(\pi) = 0 \to v(\pi,t)=0,\ \psi(\pi)=0$ 1

$\psi''(x) + \sin x = 0 \to \psi(x) = \sin x + c_1 x + c_2$ 1

$\psi(0) = 0 \Rightarrow c_2 = 0 \qquad \psi(\pi) = 0 \to c_1 = 0$

$\psi(x) = \sin x$ 1

$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$

2

$v(0,t) = 0 \qquad v(\pi,t) = 0$

$v(x,0) = -\sin x + 3\sin 2x$

1

$v(x,t) = X(x)\,T(t) \Rightarrow X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda \to X'' + \lambda X = 0,\ T' + \lambda T = 0$

$v(0,t) = 0 \to X(0) = 0$ 1 $\qquad v(\pi,t) = 0 \to X(\pi) = 0$ 1

$\lambda = \alpha^2 \to X'' + \alpha^2 X = 0 \to X(x) = B_1\cos\alpha x + B_2\sin\alpha x$

1 $X(0) = 0 \to B_1 = 0 \qquad X(\pi) = 0 \to \alpha = n$ 1

$X(x) = B_2 \sin nx \qquad T' + \lambda T = 0 \to T(t) = B_3 e^{-\lambda t} \to T(t) = B_3 e^{-n^2 t}$ 1

$\lambda = 0 \rightarrow$ Trivial solution (2) $\quad \lambda = -\alpha^2$ Trivial Solution (2)

$$V_n(x,t) = X(x)\,T(t) = A_n \sin nx \, e^{-n^2 t}$$

$$V(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-n^2 t} \quad (1)$$

$$V(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = -\sin x + 3\sin 2x$$

(2)

$A_1 = -1 \qquad A_2 = 3 \qquad A_n = 0 \qquad n \neq 1, 2$

$$V(x,t) = -\sin x \, e^{-t} + 3\sin 2x \, e^{-4t} \quad (1)$$

$$u(x,t) = V(x,t) + \psi(x) = \sin x - \sin x \, e^{-t} + 3\sin 2x \, e^{-4t} \quad (1)$$

$$u\left(\tfrac{\pi}{2}, 1\right) = 1 - e^{-1} = 1 - \frac{1}{e} \quad (1)$$

Question # 4(25 Points)

Solve following boundary value problem.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 1, \qquad 0 < y < 1$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \qquad u(x,0) = 1 + 2\cos(3\pi x)$$

$u(x,y) = X(x)\,Y(y)$ 1 $\qquad X''Y + XY'' = 0 \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$X'' + \lambda X = 0$ 1 $\qquad Y'' - \lambda Y = 0$ 1

$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \rightarrow X'(0) = 0$ 1 $\qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0 \rightarrow X'(1) = 0$ 1

$\left.\frac{\partial u}{\partial y}\right|_{y=0} \rightarrow Y'(0) = 0$ 1

1

$\lambda = 0 \rightarrow X'' = 0 \rightarrow X(x) = c_1 x + c_2$ 1

$X'(0) = 0 \rightarrow c_1 = 0$

$X'(1) = 0 \rightarrow c_1 = 0$ $\qquad \rightarrow X(x) = c_2$ 1

$Y'' = 0 \rightarrow Y(y) = c_3 y + c_4$ 1 $\qquad Y(y) = c_4$ 1

$Y'(0) = 0 \longrightarrow c_3 = 0$

$u(x,y) = X(x)\,Y(y) = c_2 c_4 = A_0$ 1

1 $\qquad$ 1

$\lambda = +\alpha^2 \rightarrow X'' + \lambda X = 0 \rightarrow X(x) = c_5 \cos\alpha x + c_6 \sin\alpha x$

1

$X'(0) = 0 \rightarrow c_6 = 0$ $\qquad X(x) = c_5 \cos n\pi x$

$X'(1) = 0 \rightarrow \sin\alpha = 0 \rightarrow \alpha = n\pi$ 1

$Y'' - \lambda Y = 0 \rightarrow Y(y) = c_7 \cosh\alpha y + c_8 \sinh\alpha y$ 1

$Y'(0) = 0 \rightarrow c_8 = 0 \rightarrow Y(y) = c_7 \cosh\alpha y$ 1

$\lambda = \alpha^2 \quad \rightarrow X(x) = c_9 \cosh \alpha x + c_{10} \sinh \alpha x \quad \begin{matrix} X'(0) = 0 \\ X'(1) = 0 \end{matrix} \rightarrow \begin{matrix} c_9 = 0 \\ c_{10} \neq 0 \end{matrix}$ Trivial Solution 3

$$u_n(x,y) = X(x)\,Y(y) = A_n \cosh(n\pi y) \cos(n\pi x)$$

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh(n\pi y) \cos(n\pi x) \qquad 1$$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x = 1 + 2\cos(3\pi x) \qquad 2$$

$A_0 = 1 \qquad A_3 = 2 \qquad A_n = 0 \quad n \neq 3$

$$u(x,y) = 1 + 2\cosh(3\pi y)\cos(3\pi x) \qquad 1$$