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MIDTERM EXAMINATION

COURSE: ENGR 311	TRANSFORM CALCULUS
INSTRUCTOR: M. ESHAGHI	MAX. MARKS: 100
DATE: 17 OCTOBER 2017	DURATION: 75 MINUTES

- *Write directly on this questions booklet. Show your work in sufficient details and clearly to qualify for partial marks*
- *DO NOT UNSTAPLE THE QUESTIONS BOOKLET*
- *YOU CAN FIND FORMULA SHEET IN THE LAST PAGE*
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Question #1 (20 Marks)

Find Inverse Laplace of following functions

a) $F(s) = e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+10s+29}\right)$ **(15 Marks)**

$$s^2+10s+29 = s^2+10s+25+4 = (s+5)^2+4$$

$$\mathcal{L}^{-1}\left\{e^{-\pi/2\, s}\left(\frac{s}{s^2+10s+29}\right)\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{s+5\ -5}{(s+5)^2+4}\right\} =$$

$$\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{s+5}{(s+5)^2+4}\right\} - 5\,\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{(s+5)^2+4}\right\} =$$

$$\left(e^{-5(t-\frac{\pi}{2})}\cos 2(t-\tfrac{\pi}{2}) - \frac{5}{2}e^{-5(t-\frac{\pi}{2})}\sin 2(t-\tfrac{\pi}{2})\right)u(t-\tfrac{\pi}{2})$$

b) $F(s) = \left(\frac{s+1}{s+2}\right)^2$ **(5 Marks)**

$$\mathcal{L}^{-1}\left\{\left(\frac{s+1}{s+2}\right)^2\right\} = \mathcal{L}^{-1}\left\{\left(\frac{s+2-1}{s+2}\right)^2\right\} = \mathcal{L}^{-1}\left\{\left(1-\frac{1}{s+2}\right)^2\right\} =$$

$$\mathcal{L}^{-1}\left\{1-\frac{2}{s+2}+\frac{1}{(s+2)^2}\right\} = \delta(t) - 2e^{-2t} + te^{-2t}$$

Question #2 (10 Marks)

Write following function in terms of unit step function and find its Laplace transform.

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t^2 & t \ge 1 \end{cases}$$

$f(t) = t^2 u(t-1)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} =$

$\mathcal{L}\{e^{-s}(t^2+2t+1)\} = \frac{2}{s^3}e^{-s} + \frac{2}{s^2}e^{-s} + \frac{1}{s}e^{-s}$

Second solution

$t^2 = (t-1)^2 - 2(t-1) + 1$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)^2 u(t-1)\} - 2\mathcal{L}\{(t-1)u(t-1)\} +$

$\mathcal{L}\{u(t-1)\} =$

$e^{-s}\frac{2}{s^3} - \frac{2}{s^2}e^{-s} + \frac{e^{-s}}{s}$

Question #3 (30 Marks)

Find Laplace Transform of following functions.

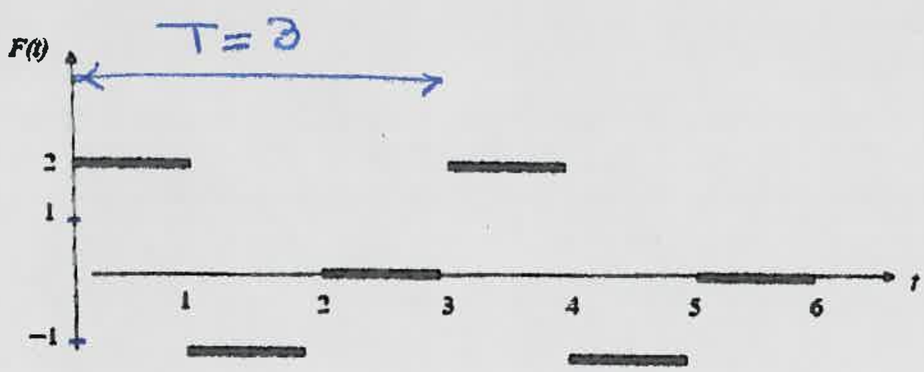
a) $f(t) = t\int_0^t \tau e^{-2\tau} d\tau$

$$\mathcal{L}\left\{t\int_0^t \tau e^{-2\tau} d\tau\right\} = -\frac{d}{ds}\mathcal{L}\left\{\int_0^t \tau e^{-2\tau} d\tau\right\} =$$

$$-\frac{d}{ds}\frac{\mathcal{L}\{t e^{-2t}\}}{s} = -\frac{d}{ds}\frac{\frac{1}{(s+2)^2}}{s} = -\frac{d}{ds}\frac{1}{s(s+2)^2} =$$

$$\frac{(s+2)^2+2s(s+2)}{(s(s+2)^2)^2}$$

b) *f(t)* is a periodic function



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-3s}}\int_0^3 e^{-st} f(t)\,dt =$$

$$\frac{1}{1-e^{-3s}}\left[2\int_0^1 e^{-st}dt + \int_1^2 e^{-st}(-1)dt + \cancel{\int_2^3 e^{-st}(0)}^{\to 0}\right] =$$

$$\frac{1}{1-e^{-3s}}\left[\left[-\frac{2}{s}e^{-st}\right]_0^1 - \left[-\frac{1}{s}e^{-st}\right]_1^2\right) = \frac{1}{1-e^{-3s}}\left(-\frac{2}{s}(e^{-s}-1)+\frac{1}{s}(e^{-2s}-e^{-s}\right.$$

Question #4 (20 Marks)

Solve following equation and find $y(t)$

$$y(t) - \int_0^t y(\tau)(t-\tau)d\tau = -\frac{1}{2}t^2$$

$$\mathcal{L}\{y\} - \mathcal{L}\{\int_0^t y(\tau)(t-\tau)d\tau\} = -\frac{1}{2}\mathcal{L}\{t^2\}$$

$$Y(s) - Y(s)\,\mathcal{L}\{t\} = -\frac{1}{2}\,\frac{2}{s^3}$$

$$Y(s)\left(1-\frac{1}{s^2}\right) = \frac{-1}{s^3} \Rightarrow$$

$$Y(s) = \frac{-1}{s(s^2-1)} \qquad y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$Y(s) = \frac{-1}{s(s^2-1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$\frac{-1}{s(s+1)(s-1)} = \frac{A(s-1)(s+1) + B(s)(s+1) + C(s)(s-1)}{s(s-1)(s+1)}$$

$s=1$: $-1 = 2B \Rightarrow B = -1/2$

$s=-1$: $-1 = 2C \Rightarrow C = -1/2$

$s=0$: $-1 = -A \rightarrow A = +1$

$$Y(s) = \frac{+1}{s} - \frac{1/2}{s-1} - \frac{\frac{1}{2}}{s+1} \Rightarrow y(t) = +1 - \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$$

Question #5 (20 Marks)

Solve following system of equation and find $x(t)$ and $y(t)$.

$$\begin{cases} \frac{d^2}{dt^2}y = \delta(t-6) + \delta(t-4) - x \\ \frac{d^2}{dt^2}y = \frac{d^2}{dt^2}x \end{cases} \qquad x(0)=0,\quad y(0)=0,\quad x'(0)=0,\quad y'(0)=0$$

$$\mathcal{L}\{y''\} = \mathcal{L}\{\delta(t-6)\} + \mathcal{L}\{\delta(t-4)\} - \mathcal{L}\{x\}$$

$$\Rightarrow s^2Y(s) - s\cancel{y(0)} - \cancel{y'(0)} = e^{-6s} + e^{-4s} - X(s)$$

$$0 \qquad 0$$

$$s^2Y(s) + X(s) = e^{-6s} + e^{-4s} \qquad (*)$$

$$\mathcal{L}\{y''\} = \mathcal{L}\{x''\} \Rightarrow s^2Y(s) - s\cancel{y(0)} - \cancel{y'(0)} = s^2X(s) - s\cancel{x(0)} - \cancel{x'(0)}$$

$$0 \quad 0 \qquad\qquad 0 \quad 0$$

$$\Rightarrow X(s) = Y(s) \qquad (*)(*)$$

plug $(*)(*)$ in $(*)$ $\Rightarrow s^2Y(s) + Y(s) = e^{-6s} + e^{-4s}$

$$\Rightarrow Y(s) = \frac{e^{-6s}}{s^2+1} + \frac{e^{-4s}}{s^2+1}$$

$$\Rightarrow y(t) = \sin(t-6)\,u(t-6) + \sin(t-4)\,u(t-4)$$

$$x(t) = y(t)$$