Question (1)

a) $$\mathcal{L}^{-1}\left\{\frac{3(s^2+s-12)+(2s-72)(s+4)^2}{(s+4)^2(s^2+s-12)}\right\} =$$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s+4)^2}+\frac{(2s-72)}{s^2+s-12}\right\} =$$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s+4)^2}+\frac{2s-72}{(s+4)(s-3)}\right\}$$

Partial Fraction

$$\frac{2s-72}{(s+4)(s-3)} = \frac{A}{(s+4)}+\frac{B}{(s-3)} = \frac{A(s-3)+B(s+4)}{(s+4)(s-3)}$$

$s = 3$: $\quad -66 = 7B \Rightarrow B = \frac{-66}{7}$

$s = -4$: $\quad -80 = -7A \Rightarrow A = \frac{-80}{-7} = \frac{80}{7}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{3}{(s+4)^2}+\frac{2s-72}{(s+4)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{(s+4)^2}+\frac{80}{7}\frac{1}{s+4}-\frac{66}{7}\frac{1}{s-3}\right\}$$

$$\Rightarrow f(t) = 3e^{-4t}\,t+\frac{80}{7}e^{-4t}-\frac{66}{7}e^{3t}$$

Question 1

b) $\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+10s+29}\right)\right\}$

$s^2+10s+29$ $\quad b^2-4ac = 100-4(29) < 0$

No real root

$s^2+10s+29 = s^2+10s+25+4 = (s+5)^2+4$

$$\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+10s+29}\right)\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{(s+5)-5}{(s+5)^2+4}\right\}_s$$

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s+5)^2+4}e^{-\frac{\pi}{2}s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^2+4}e^{-\pi/2 s}\right\}_s$$

$$e^{-5(t-\frac{\pi}{2})}\cos 2(t-\tfrac{\pi}{2})\,\mathcal{U}(t-\tfrac{\pi}{2}) - \frac{5}{2}e^{-5(t-\frac{\pi}{2})}\sin 2(t-\tfrac{\pi}{2})\,\mathcal{U}(t-\tfrac{\pi}{2})$$

In this problem, you have to use first and second translation theorems.

Question 1

c) $\mathcal{L}^{-1}\left\{\dfrac{8}{s^3(s^2-s-2)}\right\} \Rightarrow$

$(s^2-s-2) = (s-2)(s+1)$

$$\mathcal{L}^{-1}\left\{\frac{8}{s^3(s-2)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-2}+\frac{E}{s+1}\right\}$$

$$\frac{8}{s^3(s-2)(s+1)} = \frac{A(s^2)(s-2)(s+1)+B(s)(s-2)(s+1)+C(s-2)(s+1)+D(s^3)(s+1)+E(s^3)(s-2)}{s^3(s-2)(s+1)}$$

$s=0 \quad : \quad 8 = -2C \Rightarrow C = -4$

$s=2 \quad : \quad 8 = 24D \Rightarrow D = 1/3$

$s=-1 \quad : \quad 8 = -3C \Rightarrow E = +8/3$

coefficient of s^4 : $\quad 0 = A + D + E \Rightarrow A = -3$

" " s^3 : $\quad 0 = -A + B + D - 2E \Rightarrow B = 2$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{8}{s^3(s^2-s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3}{s}+\frac{2}{s^2}-\frac{4}{s^3}+\frac{1/3}{s-2}+\frac{8/3}{s+1}\right\}$$

$$= -3+2t-2t^2+\tfrac{1}{3}e^{2t}+\tfrac{8}{3}e^{-t}$$

Question 2

a) $\mathcal{L}\{e^{-2t}\, t \sinh t\} = -\frac{d}{ds}\mathcal{L}\{e^{-2t}\sinh t\} =$

$$-\frac{d}{ds}\mathcal{L}\{\sinh t\}_{s\to s+2} =$$

$$-\frac{d}{ds}\frac{1}{(s+2)^2-1} = \frac{2(s+2)}{((s+2)^2-1)^2}$$

b) $\mathcal{L}\{(t-3)\sin 2t\} = \mathcal{L}\{t\sin 2t\} - \mathcal{L}\{3\sin 2t\} =$

$$-\frac{d}{ds}\mathcal{L}\{\sin 2t\} - 3\,\frac{2}{s^2+4} =$$

$$-\frac{d}{ds}\frac{2}{s^2+4} - \frac{6}{s^2+4} =$$

$$\frac{+4s}{(s^2+4)^2} - \frac{6}{s^2+4}$$

Question 8 :

a) $\mathcal{L}\{\int_0^t y(x)\,y(t-x)\,dx\} = \mathcal{L}\{6t^3\}$

$Y(s)\,Y(s) = 6\,\frac{3!}{s^4} = \frac{36}{s^4}$

$\Rightarrow Y^2(s) = \frac{36}{s^4} \Rightarrow Y(s) = \frac{6}{s^2}$

$y(t) = 6t$

b) $\mathcal{L}\{\int_0^t y(u)\,y(t-u)\,du\} = \frac{1}{2}\mathcal{L}\{\sin t\} - \frac{1}{2}\mathcal{L}\{t\cos t\}$

$Y(s)\,Y(s) = \frac{1}{2}\,\frac{1}{s^2+1} - \frac{1}{2}\left(-\frac{d}{ds}\,\frac{s}{s^2+1}\right) \Rightarrow$

$Y^2(s) = \frac{1}{2}\,\frac{1}{s^2+1} + \frac{1}{2}\,\frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{1}{2}\,\frac{1}{s^2+1} + \frac{1}{2}\,\frac{-s^2+1}{(s^2+1)^2}$

$Y^2(s) = \frac{1}{2}\,\frac{(s^2+1)-s^2+1}{(s^2+1)^2} = \frac{1}{2}\,\frac{2}{(s^2+1)^2} = \frac{1}{(s^2+1)^2}$

$\Rightarrow Y(s) = \frac{1}{(s^2+1)} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \sin t$

$y(t) = \sin t$

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Question :

$$\mathcal{L}y'' + 10\mathcal{L}y' + 9\mathcal{L}y = \mathcal{L}f(t)$$

$$f(t) = 1\{u(t-5) - u(t-7)\}$$

$$\mathcal{L}\{f(t)\} = \left(\frac{e^{-5s} - e^{-7s}}{s}\right)$$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 10(sY(s) - \cancel{y(0)}^{0}) + 9Y(s) = \mathcal{L}\{f(t)\}$$

$$(s^2 + 10s + 9)Y(s) = \frac{1}{s}(e^{-5s} - e^{-7s})$$

$$Y(s) = \frac{1}{(s+9)(s+1)s}(e^{-5s} - e^{-7s})$$

$$\frac{1}{s(s+1)(s+9)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+9} = \frac{A(s+1)(s+9) + B(s)(s+9) + C(s)(s+1)}{s(s+1)(s+9)}$$

$$\Rightarrow 1 = A(s+1)(s+9) + B(s)(s+9) + C(s)(s+1)$$

$$s = -1 \Longrightarrow \quad 1 = -8B \rightarrow B = -\frac{1}{8}$$

$$s = -9 \Longrightarrow \quad 1 = 72C \rightarrow C = \frac{1}{72}$$

$$s = 0 \Longrightarrow \quad 1 = 9A \rightarrow A = 1/9$$

$$\Rightarrow Y(s) = \left(\frac{1}{9}\frac{1}{s} - \frac{1}{8}\frac{1}{s+1} + \frac{1}{72}\frac{1}{s+9}\right)e^{-5s} - \left(\frac{1}{9}\frac{1}{s} - \frac{1}{8}\frac{1}{s+1} + \frac{1}{72}\frac{1}{s+9}\right)e^{-7s}$$

$$y(t) = \left(\frac{1}{9} - \frac{1}{8}e^{-(t-5)} + \frac{1}{72}e^{-9(t-5)}\right)u(t-5) - \left(\frac{1}{9} - \frac{1}{8}e^{-(t-7)} + \frac{1}{72}e^{-9(t-7)}\right)u(t-7)$$