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MIDTERM EXAMINATION

COURSE: ENGR 311
INSTRUCTOR: M. ESHAGHI
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TRANSFORM CALCULUS
MAX. MARKS: 100
DURATION: 75 MINUTES

- *Write directly on this questions booklet. Show your work in sufficient details and clearly to qualify for partial marks*
- *DO NOT UNSTAPLE THE QUESTIONS BOOKLET*
- *YOU CAN FIND FORMULA SHEET IN THE LAST PAGE*
- *TOTAL NUMBER OF PAGES: 7*

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Question #1 (20 Marks)

Solve following integral equation and find $f(t)$

$$\int_0^t f(\tau)\, f(t-\tau)\,d\tau = \frac{1}{2}(\sin(t) - t\cos(t))$$

$$\mathcal{L}\left\{\int_0^t f(\tau)f(t-\tau)\,d\tau\right\} = \frac{1}{2}\mathcal{L}\{\sin t\} - \frac{1}{2}\mathcal{L}\{t\cos t\}$$

$$F(s)\;F(s) = \frac{1}{2}\;\frac{1}{s^2+1} - \frac{1}{2}\mathcal{L}\{t\cos t\}$$

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\;\frac{s}{s^2+1} = -\frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

$$\Rightarrow (F(s))^2 = \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right) = \frac{1}{(s^2+1)^2}$$

$$\Rightarrow F(s) = \frac{1}{s^2+1} \quad \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \sin t$$

Question #2 (25 Marks)

Solve following equation and find $y(t)$

$y' + y = e^{-3t}\cos(2t)$ $\qquad\qquad$ $y(0) = 0$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\cos 2t\}$$

$$sY(s) - \underbrace{\cancel{y(0)}}_{0} + Y(s) = \frac{s+3}{(s+3)^2+4}$$

$$Y(s) = \frac{s+3}{(s+1)\left((s+3)^2+4\right)} = \frac{s+3}{(s+1)(s^2+6s+13)}$$

$$\frac{s+3}{(s+1)(s^2+6s+13)} = \frac{A}{(s+1)} + \frac{Bs+C}{s^2+6s+13}$$

$$\Longrightarrow s+3 = A(s^2+6s+13) + (Bs+C)(s+1)$$

coefficient of s^2 : $\quad 0 = A + B$

" $\quad$ " $\quad s$: $\quad 1 = 6A + B + C$

" $\quad$ " $\quad s$; $\quad 3 = 13A + C$

$$A = 1/4 \qquad B = -1/4 \qquad C = -1/4$$

$$\Rightarrow Y(s) = \frac{1/4}{(s+1)} - \frac{1}{4}\,\frac{s}{(s+3)^2+4} + \frac{1}{4}\,\frac{1}{(s+3)^2+4}$$

$$\Rightarrow Y(s) = \frac{1/4}{(s+1)} - \frac{1}{4}\,\frac{s+3}{(s+3)^2+4} + \frac{1}{2}\,\frac{1}{(s+3)^2+4}$$

$$y(t) = 1/4\, e^{-t} - \frac{1}{4}\, e^{-3t}\cos 2(t) + \frac{1}{4}\, e^{-3t}\sin 2(t)$$

Question #3 (20 Marks)

Find Inverse Laplace of following functions

a) $F(s) = \frac{s+\pi}{s^2+\pi^2}e^{-s}$

$$\mathcal{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}e^{-s}\right\} + \mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}e^{-s}\right\} =$$

$$\cos\pi(t-1)\,u(t-1) + \sin\pi(t-1)\,u(t-1)$$

b) $F(s) = \frac{\sqrt{\pi}}{s^{3.5}}$

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{7/2}}\right\} = \frac{\sqrt{\pi}}{\Gamma(7/2)}\mathcal{L}^{-1}\left\{\frac{\Gamma(5/2+1)}{s^{5/2+1}}\right\} =$$

$$\frac{\sqrt{\pi}}{\Gamma(7/2)}t^{2.5} = \frac{\sqrt{\pi}}{\frac{7}{2}\Gamma(3/2+1)}t^{2.5} = \frac{\sqrt{\pi}}{\frac{5}{2}\,\frac{3}{2}\,\Gamma(\frac{1}{2}+1)}t^{2.5} =$$

$$\frac{\cancel{\sqrt{\pi}}}{\frac{5}{2}\times\frac{3}{2}\,\frac{1}{2}\,\cancel{\sqrt{\pi}}}t^{2.5} = \frac{8}{15}t^{2.5}$$

Question #4 (20 Marks)

Find Laplace Transform of following functions.

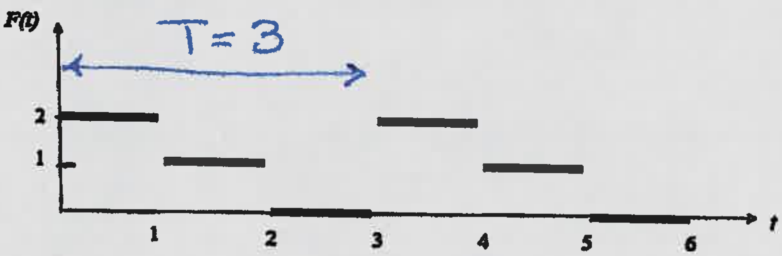
a) $f(t) = \int_0^t \cos(t-\tau)e^{\tau}\sin(\tau)d\tau$

$$\mathcal{L}\left\{\int_0^t \cos(t-\tau)\, e^{\tau} \sin(\tau)\, d\tau\right\} =$$

$$\mathcal{L}\{\cos(t)\} \quad \mathcal{L}\{e^t \sin t\} =$$

$$\frac{s}{s^2+1} \times \frac{1}{(s-1)^2+1} = \frac{s}{(s^2+1)((s^2-1)+1)}$$

b) *f(t)* is a periodic function



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st} f(t)\, dt =$$

$$\frac{1}{1-e^{-3s}}\left(\int_0^1 2e^{-st}\,dt + \int_1^2 e^{-st}\,dt + \cancel{\int_2^3 e^{-st}(0)\,dt}\right) =$$

0

$$\frac{1}{1-e^{-3s}}\left(\left(\frac{-2}{s}e^{-st}\right)_0^1 + \left(\frac{-1}{s}e^{-st}\right)_1^2\right) = \frac{1}{1-e^{-3s}}\left(-\frac{2}{s}(e^{-s}-1) - \frac{1}{s}(e^{-2s} - e\right.$$

Question #5 (15 Marks)

Solve following system of equation and find $x(t)$ and $y(t)$.

$$\begin{cases} \frac{d^2}{dt^2}y + x = \delta(t-3) \\ \frac{dy}{dt} - \frac{dx}{dt} = 0 \end{cases} \qquad x(0)=0, \quad y(0)=0, \quad y'(0)=0$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{x\} = \mathcal{L}\{\delta(t-3)\}$$

$$s^2Y(s) - \cancel{s\,y(0)} - \cancel{y'(0)} + X(s) = e^{-3s}$$

0 0

$$s^2 Y(s) + X(s) = e^{-3s} \qquad (x)$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{x'\} = 0 \Rightarrow sY(s) - sX(s) = 0$$

$$X(s) = Y(s) \qquad (x)(x)$$

Plug (x)(x) in (x)

$$s^2 Y(s) + Y(s) = e^{-3s} \Rightarrow Y(s)(s^2+1) = e^{-3s}$$

$$Y(s) = \frac{e^{-3s}}{s^2+1} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \sin(t-3)u(t-3)$$

Function	Laplace Transform
1	$1/s$
t^n	$n!/s^{n+1}$
e^{at}	$1/(s-a)$
$\sin kt$	$k/(s^2+k^2)$
$\cos kt$	$s/(s^2+k^2)$
$\sinh kt$	$k/(s^2-k^2)$
$\cosh kt$	$s/(s^2-k^2)$
$t^\alpha \quad \alpha > -1$	$\Gamma(\alpha+1)/s^{\alpha+1} \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
$\Gamma(\frac{1}{2})$	$\sqrt{\pi}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)\,\mathscr{U}(t-a)$	$e^{-as}F(s)$
$g(t)\,\mathscr{U}(t-a)$	$e^{-as}\mathscr{L}\{g(t+a)\}$
$t^n f(t)$	$(-1)^n d^n F(s)/ds^n$
$f*g$	$F(s).G(s)$
$\int_0^t f(\tau)\,d\tau$	$F(s)/s$
$f(t)$ where $f(t+T)=f(t)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$
$\delta(t-t_0)$	e^{-st_0}

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin A \cdot \sin B = \frac{1}{2}\left[-\cos(A+B) + \cos(A-B)\right]$$

$$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \cdot \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$