Solution to Test 1 (version A)

MAT1300D, Fall 2019
Total = 20 marks

Part I. Multiple-choice Questions (2 × 5 = 10 marks)

BAFCC

1. Suppose some values of functions $f(x)$ and $g(x)$ are given in the following table:

x	1	2	3
$f(x)$	3	2	1
$g(x)$	2	3	1

Then $(f \circ g)(2) + g^{-1}(2) =$

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5; (F) 6.

Solution. (B) $(f \circ g)(2) = f(g(2)) = f(3) = 1$. $g^{-1}(2) = 1$. $(f \circ g)(2) + g^{-1}(2) = 2$.

2. The root of the equation $3^{2x-1} = 5^x$, expressed in the natural logarithm is

(A) $\dfrac{\ln 3}{\ln(9/5)}$; (B) $\dfrac{\ln 3}{\ln(3/5)}$; (C) $\dfrac{\ln(9/5)}{\ln 3}$;
(D) $\dfrac{\ln(3/5)}{\ln 3}$; (E) $\dfrac{\ln 5}{\ln(9/5)}$; (F) $\dfrac{\ln(9/5)}{\ln 5}$.

Solution. (A) Take natural logarithm on both sides. Then $(2x - 1)\ln 3 = x \ln 5$. $x(2\ln 3 - \ln 5) = \ln 3$. $x = \dfrac{\ln 3}{2\ln 3 - \ln 5} = \dfrac{\ln 3}{\ln 3^2 - \ln 5} = \dfrac{\ln 3}{\ln(9/5)}$.

3. The equation of the tangent line of the graph of the function $f(x) = \dfrac{x}{e^{2(x+1)}}$ at the point $(-1, -1)$ is

(A) $3x - y + 4 = 0$; (B) $2x - y + 1 = 0$; (C) $x + 3y + 4 = 0$;
(D) $x - 3y - 2 = 0$; (E) $2x + y - 3 = 0$; (F) $3x - y + 2 = 0$.

Solution. (F) The derivative of this function is

$f'(x) = \dfrac{e^{2(x+1)} - 2xe^{2(x+1)}}{e^{2(x+1)}} = \dfrac{1-2x}{e^{2(x+1)}}$. When $x = -1$, $f'(-1) = 3$. The equation of the tangent line is $y = 3(x + 1) - 1$, or $3x - y + 2 = 0$.

4. Let $f(x) = \sqrt{x^2+9}$. Then $f'(4) =$

(A) 1; (B) $\sqrt{5}$; (C) $\frac{4}{5}$; (D) $\frac{4}{\sqrt{5}}$; (E) $2\sqrt{5}$; (F) 5.

Solution. (C) Use the chain rule. Let $u = x^2 + 9$. Then $y = \sqrt{u}$.

$f'(x) = y_u' u_x' = \left(\frac{1}{2\sqrt{u}}\right)(2x) = \frac{x}{\sqrt{x^2+9}}$. Hence, $f'(4) = \frac{4}{\sqrt{4^2+9}} = \frac{4}{5}$.

5. Suppose an amount $10000 is deposit to an account with annual interest rate 3% compounded continuously. How many years would it take to reach a balance 11000?

(A) 0.03 ln 1.1; (B) $\frac{0.03}{\ln 1.1}$; (C) $\frac{\ln 1.1}{0.03}$;

(D) $\frac{\ln 0.03}{\ln 1.1}$; (E) 1.1 ln 0.03; (F) $\frac{\ln 1.1}{\ln 0.03}$.

Solution. (C) The balance after t years is $A(t) = 10000e^{0.03t}$. Let $11000 = 10000e^{0.03t}$. Then $e^{0.03t} = 1.1$. $0.03t = \ln 1.1$. $t = \frac{\ln 1.1}{0.03}$.

Part II. Detailed-answer Questions (10 marks)

1. (3 marks) Find the vertical asymptote(s) and horizontal asymptote(s) of the graph of the function $f(x) = \frac{x^2}{3x^2 - 2x - 1}$.

Answer. The vertical asymptote(s) is/are $\underline{x = 1,\ x = -1/3}$.

The horizontal asymptote(s) is/are $\underline{y = 1/3}$.

Show your work: Let $3x^2 - 2x - 1 = 0$. $x = 1$, $x = -\frac{1}{3}$. Since the numerator is not 0 at these points, we have vertical asymptotes $x = 1$, $x = -\frac{1}{3}$.

Since $\lim_{x\to\infty} \frac{x^2}{3x^2 - 2x - 1} = \lim_{x\to-\infty} \frac{x^2}{3x^2 - 2x - 1} = \frac{1}{3}$, this function has a horizontal asymptote $y = \frac{1}{3}$.

2. (2 marks) For which value of a and b, is the function

$$f(x) = \begin{cases} x+a, & x<-1 \\ 2b-x, & -1 \le x \le 1 \\ 2ax+3b, & x>1 \end{cases}$$

continuous for all real numbers *x*?

Answer. This function is continuous for all real number *x* when a = __0__ and b = __−1___ .

Work. This function is linear and continuous when $x < -1$, $-1 < x < 1$, or $x > 1$. It is continuous at $x = -1$ if

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(x+a) = a-1 = \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(2b-x) = 2b+1.$$

It is continuous at $x = 1$ if

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2b-x) = 2b-1 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2ax+3b) = 2a+3b.$$

Then we have a system of equations:

$a - 1 = 2b + 1,$
$2b - 1 = 2a + 3b.$

Solving this system, we have $a = 0$, $b = -1$.

3. (5 marks) Use the definition of the derivative to find the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$ at $x = 1$.

Solution. $f'(1) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{1+h}} - \frac{1}{\sqrt{1}}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{1+h}} - 1\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1-\sqrt{1+h}}{\sqrt{1+h}}\right)$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(1-\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)}{\sqrt{1+h}\left(1+\sqrt{1+h}\right)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1-(1+h)}{\sqrt{1+h}\left(1+\sqrt{1+h}\right)}\right) = -\lim_{h\to 0}\frac{1}{\sqrt{1+h}\left(1+\sqrt{1+h}\right)} = -\frac{1}{2}.$$