COMP 232 Assignment 4 Solution

Question 1

Base case:
$n = 18$, $18 = 7 + 7 + 4$;
$n = 19$, $19 = 7 + 4 + 4 + 4$;
$n = 20$, $20 = 4 + 4 + 4 + 4 + 4$;
$n = 21$, $21 = 7 + 7 + 7$.
Inductive hypothesis: assume when $18 \leq n \leq k$, n is a sum of 4's and 7's.
Inductive step: we need to show when $n = k + 1$, the statement is true.
From IH, we know $n = k - 3$ is a sum of 4's and 7's. Thus, $n = k - 3 + 4 = k + 1$ is also a sum of 4's and 7's.
Thus, the statement is true for all $n \geq 18$.

Question 2

1. $S = \{(a, b) \ : \ a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a + b \text{ is odd}\}$ Solution:
    - Base case 1: $(1, 2) \in S$.
    - Base case 2: $(2, 1) \in S$.
    - Recursive case 1:if $(i, j) \in S$ then $(i + 1, j + 1) \in S$.
    - Recursive case 2: if $(i, j) \in S$ then $(i, j + 2) \in S$.
    - Recursive case 3: if $(i, j) \in S$ then $(i + 2, j) \in S$.
2. $S = \{(a, b) \ : \ a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a|b\}$ Solution:
    - Base case: $(n, n) \in S$ for all $n \in \mathbb{Z}^+$.
    - Recursive case: if $(i, j) \in S$ then $(i, j \cdot k) \in S$, for all $k \in \mathbb{Z}^+$.
3. $S = \{(a, b) \ : \ a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } 3|(a + b)\}$ Solution:
    - Base case 1: $(1, 2) \in S$.
    - Base case 2: $(2, 1) \in S$.
    - Recursive case 1: if $(i, j) \in S$ then $(i + 3, j) \in S$.
    - Recursive case 2: if $(i, j) \in S$ then $(i, j + 3) \in S$.

- Recursive case 3: if $(i, j) \in S$ then $(i+1, j+2) \in S$.
- Recursive case 4: if $(i, j) \in S$ then $(i+2, j+1) \in S$.

Question 3

Base case: $n = 1$, $LHS = 1 \cdot 1! = 1$, $RHS = (1+1)! - 1 = 2 - 1 = 1$. Thus, $LHS = RHS$.
Inductive hypothesis: Assume when $n = k$, the statement $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$ is true.
Inductive step: We need to show $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$

$$\begin{aligned} LHS &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 \\ &= (k+2)! - 1 \\ &= RHS \end{aligned}$$

Thus, the statement is true for all $n \geq 1$.

Question 4

Base case: when $n = 1$, $4^{1+1} + 5^{2-1} = 21 \equiv 0 \mod 21$.
Inductive hypothesis: assume when $n = k$, $4^{k+1} + 5^{2k-1} \equiv 0 \mod 21$.
Inductive step:
By inductive hypothesis:

$$\begin{aligned} 4^{k+1} + 5^{2k-1} &\equiv 0 \mod 21 \\ 4^{k+1} + 5^{2k-1} &= 21a \\ 5^{2k-1} &= 21a - 4^{k+1} \end{aligned}$$

The equivalence we want to show:

$$\begin{aligned} &4^{k+1+1} + 5^{2(k+1)-1} \\ =&4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1} \\ =&4 \cdot 4^{k+1} + 25(21a - 4^{k+1}) \text{ by substitution} \\ =&21 \cdot 4^{k+1} + 25 \cdot 21a \\ =&21(4^{k+1} + 25a) \\ \equiv&0 \mod 21 \end{aligned}$$

Question 5

Base case:
when $n = 1$, $f_{4n} = f_4 = 3 \equiv 0 \mod 3$.

Inductive hypothesis: assume $n = k$, $f_{4n} \equiv 0 \mod 3$.
Inductive step: when $n = k + 1$

$$
\begin{aligned}
f_{4k+4} &= f_{4k+3} + f_{4k+2} \\
&= f_{4k+2} + 2f_{4k+1} + f_{4k} \\
&= f_{4k+1} + 2f_{4k-1} + 4f_{4k} \\
&= 5f_{4k} + 3f_{4k-1}
\end{aligned}
$$

From inductive hypothesis, we know $5f_{4k} \equiv 0 \mod 3$. And $3f_{4k-1} \equiv 0 \mod 3$.
Thus, $f_{4k+4} \equiv 0 \mod 3$.

Question 6

Base case: when $n = 2$, $LHS = f_{2-1}f_{2+1} - f_2^2 = 1 \times 2 - 1 = 1 = 1^2 = RHS$.
Inductive hypothesis: assume when $n = k$, $f_{k-1}f_{k+1} - f_k^2 = (-1)^k$.
Inductive step: when $n = k + 1$,

$$
\begin{aligned}
f_k f_{k+2} - f_{k+1}^2 &= f_k(f_k + f_{k+1}) - f_{k+1}^2 \\
&= f_k^2 + f_k f_{k+1} - f_{k+1}^2 \\
&= f_k^2 - f_{k+1}(f_{k+1} - f_k) \\
&= f_k^2 - f_{k+1}(f_k + f_{k-1} - f_k) \\
&= f_k^2 - f_{k+1}f_{k-1} \\
&= -(-1)^k \\
&= (-1)^{k+1}
\end{aligned}
$$

Thus, the statement is true for all $n \geq 2$.

Question 7

	Reflexive	Symmetric	Antisymmetric	Transitive
a	No	No	Yes	No
b	No	No	Yes	No
c	Yes	Yes	No	Yes
d	No	Yes	No	No
e	Yes	Yes	No	No
f	No	Yes	No	No
g	Yes	No	Yes	Yes
h	No	Yes	No	No

Question 8

a. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Question 9

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Question 10

$$\begin{aligned} [1] &= \{1, 7, 11, 13\} \\ [2] &= \{2, 4, 8, 14\} \\ [3] &= \{3, 9\} \\ [5] &= \{5\} \\ [6] &= \{6, 12\} \\ [10] &= \{10\} \\ [15] &= \{15\} \end{aligned}$$

Question 11

a) Yes

b) Yes

c) Yes

d) No

Question 12

a) Yes

b) No

c) Yes

d) No