Department of Computer Science & Software Engineering

Comp232 Mathematics for Computer Science

Winter 2017

Assignment 3. Solutions.

1. For each of the following, determine whether it is valid or invalid. If valid then give a proof. If invalid then give a counter example.

   (a) $A \subseteq B \;\Rightarrow\; \overline{B} \subseteq \overline{A}$.

   Solution: Valid.
   Assume $A \subseteq B$, and let $x \in \overline{B}$. Now $x \in A$ or $x \notin A$. If $x \in A$, since $A \subseteq B$ we get $x \in B$, a contradiction. Therefore it must be that $x \notin A$, that is, $x \in \overline{A}$.

   (b) $B \cap C \subseteq A \;\Rightarrow\; (C - A) \cap (B - A) = \varnothing$.

   Solution: Valid.
   Proof by contradiction: Suppose $B \cap C \subseteq A$, and $(C - A) \cap (B - A) \neq \varnothing$. Then there exists an element $x \in (C - A) \cap (B - A)$. Thus this x satisfies $x \in C$, $x \in B$ and $x \notin A$. Thus $x \in B \cap C$, and since $B \cap C \subseteq A$, we get $x \in A$; a contradiction.

2. Let $f : \mathbb{R} \to \mathbb{Z}$, where $f(x) = \lceil 2x - 1 \rceil$.

   (a) Find $f(A)$, where $A = \{x \in \mathbb{R} \mid 1 \le x \le 4\}$.

   Solution: $f(A) = \{1, 2, 3, 4, 5, 6, 7\} = \{x \in \mathbb{Z} \mid 1 \le x \le 7\}$.

   (b) Find $f^{-1}(B)$, where $B = \{-9, -8\}$.

   Solution: $f^{-1}(B) = \{x \in \mathbb{R} \mid -9 \le x < -7\}$.

3. Give an example of a function

   (a) $f : \mathbb{Z} \to \mathbb{N}$ that is both 1-1 and onto.

   Solution:

$$f(n) = \begin{cases} 2n & \text{if } n \ge 0 \\ -(2n+1) & \text{if } n < 0 \end{cases}$$

   (b) $f : \mathbb{N} \to \mathbb{Z}$ that is both 1-1 and onto.

   Solution:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

4. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined as $f(m,n) = (3m + 7n, 2m + 5n)$. Is f a bijection, i.e., one-to-one and onto? If yes then give a formal proof, based on the definitions of one-to-one and onto, and derive a formula for f^{-1}. If no then explain why not.

Solution: The function f is a bijection.

Proof: To see that f is an injection, suppose $f(m_1, n_1) = f(m_2, n_2)$, that is,

$$3m_1 + 7n_1 = 3m_2 + 7n_2 \quad (1)$$
$$2m_1 + 5n_1 = 2m_2 + 5n_2 \quad (2)$$

Multiply Eq. (1) by 2 and Eq. (2) by 3, yielding

$$6m_1 + 14n_1 = 6m_2 + 14n_2 \quad (3)$$
$$6m_1 + 15n_1 = 6m_2 + 15n_2 \quad (4)$$

and then subtract Eq. (3) from Eq. (4), yielding $n_1 = n_2$. Substituting $n_1 = n_2$ in Equation (1) yields $m_1 = m_2$.

To see that f is a surjection, let $(k, \ell) \in \mathbb{Z} \times \mathbb{Z}$. Then

$$3m + 7n = k \quad (5)$$
$$2m + 5n = \ell \quad (6)$$

Multiplying Eq. (5) by 2 and Eq. (6) by 3 yields

$$6m + 14n = 2k \quad (7)$$
$$6m + 15n = 3\ell \quad (8)$$

Subtracting Eq. (7) from Eq. (8) yields

$$n = 3\ell - 2k \quad (9)$$

Multiplying Eq. (5) by 3 and Eq. (6) by 4 yields

$$9m + 21n = 3k \quad (10)$$
$$8m + 20n = 4\ell \quad (11)$$

Subtracting (11) from (10) yields

$$m + n = 3k - 3\ell \quad (12)$$

Substituting (9) in (12) yields

$$m = 5k - 6\ell \quad (13)$$

We can thus see that for each pair $(k, \ell) \in \mathbb{Z} \times \mathbb{Z}$ there is a pair $(5k - 6\ell, 3\ell - 2k) \in \mathbb{Z} \times \mathbb{Z}$ such that, $f((5k - 6\ell, 3\ell - 2k)) = (k, \ell)$. So, f is a surjective function.

After proving f is surjective, deriving a formula for f^{-1} is not difficult anymore. Here we have: $f^{-1}(k, \ell) = (5k - 6\ell, 3\ell - 2k)$.

5. (a) Suppose that Hilberts Grand Hotel is fully occupied, but the hotel closes all even numbered rooms for maintenance. Show that all guests can remain in the hotel.

   Solution: Move guest in room n to room $2n - 1$.

   (b) Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

   Solution: First move old guest from room n to room $2n - 1$. Then all even-numbered rooms are free. Now you can put new guest n in room $2n$.

6. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

   Solution: Yes.

   Proof by contradiction: Suppose that f and $f \circ g$ are one-to-one, and that g is not one-to-one. Then there are $a, b \in A$, such that $a \neq b$ and $g(a) = g(b)$. Since f is one-to-one, we have $f(g(a)) = f(g(b))$. But $f \circ g(a) = f(g(a)) = f(g(b)) = f \circ g(b)$, contradicting the assumption that $f \circ g$ is one-to-one.

7. Let $A = \{1, 2, 3, 4, 5\}$.

   (a) How many total functions $f : A \to A$ are there?

   Solution:

   Each of the five elements in A can be mapped (independently of how the other elements in A are mapped) to any of the five elements in A. Consequently, there are $5^5 = 3125$ possible functions $f : A \to A$.

   (b) How many of the functions in (a) are one-to-one?

   Solution:

   Suppose a function f maps 1 to any of the five elements in A. Then there are four choiches left for mapping 2, three choiches for 3, two choiches for 4, and one choiche for 5. Consequently, each of the $5! = 120$ permutations of $1, 2, 3, 4, 5$ represents a one-to-one function on A.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = 2x + 1$ and $g \circ f(x) = 2x + 11$. Find the rule for f.

Solution:

We know that $g(x) = 2x + 1$, and $g \circ f(x) = 2x + 11$. Let $f(x) = ax + b$, then we have

$$g \circ f(x) = g(f(x)) = 2(ax + b) + 1,$$

so we have

$$2(ax + b) + 1 = 2x + 11$$

consequently,

$$2(ax + b) + 1 = 2ax + 2b + 1 = 2x + 11$$

which implies that

$$2a = 2 \Rightarrow a = 1$$

$$2b + 1 = 11 \Rightarrow b = 5$$

finally we have

$$f(x) = x + 5.$$

9. Prove or disprove the statements below.

(a) For all positive real numbers x and y, $\lfloor x \cdot y \rfloor \leq \lfloor x \rfloor \cdot \lfloor y \rfloor$.

Solution: The claim is false.
Counter-example: $x = y = 1.5$. Then $\lfloor x \cdot y \rfloor = \lfloor 1.5 \cdot 1.5 \rfloor = \lfloor 2.25 \rfloor = 2$, and $\lfloor x \rfloor \cdot \lfloor y \rfloor = \lfloor 1.5 \rfloor \cdot \lfloor 1.5 \rfloor = 1 \cdot 1 = 1$.

(b) For all positive real numbers x and y, $\lceil x \cdot y \rceil \leq \lceil x \rceil \cdot \lceil y \rceil$.

Solution: The claim is true.
Proof: We have $x \leq \lceil x \rceil$ and $y \leq \lceil y \rceil$, and thus $x \cdot y \leq \lceil x \rceil \cdot \lceil y \rceil$. Consequently $\lceil x \cdot y \rceil \leq \big\lceil \lceil x \rceil \cdot \lceil y \rceil \big\rceil = \lceil x \rceil \cdot \lceil y \rceil$.

10. (a) Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, where a, b, c, d, and m are integers with $m \geq 2$, then $a - c \equiv b - d \pmod{m}$.

Solution: $a \equiv b \pmod{m}$ means that $\exists k \in \mathbb{Z}$, such that $a = b + km$. Likewise, $c = d + k'm$, for some $k' \in \mathbb{Z}$. Thus $a - c = b - d + km - k'm = b - d + (k - k')m$, which means that $a - c \equiv b - d \pmod{m}$.

(b) Show that if am b, k and m are integers such that $k \geq 1$, $m \geq 2$, and $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$

Solution: Since $a \equiv b \pmod{m}$, we have

$$a = b + p_1 \cdot m,$$

for some $p_1 \in \mathbb{Z}$. We shall show by an induction on k that for each k there is a $p_k \in \mathbb{Z}$, such that $a^k = b^k + p_k \cdot m$. The claim then follows.

The base case $k = 1$ is the assumption. For the inductive hypothesis, suppose the claim holds for some k, that is

$$a^k = b^k + p_k \cdot m$$

for some $p_k \in \mathbb{Z}$. We need to show that it holds for $k + 1$. We have

$$\begin{aligned} a \cdot a^k &= \\ (b + p_1 \cdot m)(b^k + p_k \cdot m) &= \\ b^{k+1} + b \cdot p_k \cdot m + p_1 \cdot m \cdot b^k + p_1 \cdot m \cdot p_k \cdot m &= \\ b^{k+1} + (b \cdot p_k + p_1 \cdot b^k + p_1 \cdot m \cdot p_k) \cdot m & \end{aligned}$$

In other words,

$$a^{k+1} = b^{k+1} + \underbrace{(b \cdot p_k + p_1 \cdot b^k + p_1 \cdot m \cdot p_k)}_{p_{k+1}} \cdot m.$$