COMP 232 Assignment 1 Solution

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Question 1

For each of the arguments below, formalize them in propositional logic. If the argument is valid identify which inference rule was used, and formulate the tautology underlying the rule. If the argument is invalid, state whether the inverse or converse error was made.

a) All cheater sit in the back row.
George sits in the back row.
George is a cheater.
Answer:
x consists of all the students. $C(x)$: x is a cheater. $B(x)$: x sits in the back row. Argument is in the form.

$$\begin{array}{ll} & \forall x(C(x) \to B(x)) \\ & B(George) \\ \hline \therefore & C(George) \end{array}$$

Invalid, converse error.

b) For all students x, if x studies discrete math, then x is good at logic.
Dawn studies discrete math.
Dawn is good at logic.
Answer:
$D(x)$: x studies discrete math. $L(x)$: x is good at logic. Argument is in the form.

$$\begin{array}{ll} & \forall x(D(x) \to L(x)) \\ & D(Dawn) \\ \hline \therefore & L(Dawn) \end{array}$$

Valid, universal instantiation and modus ponens.

c) If the compilation of a computer program produces error messages, then the program is not correct or the compiler is faulty.
The compilation of this program does not produce error messages.
This program is correct and the compiler is not faulty.
Answer:
e: produces error message. p: program is correct. c: the compiler is working. Argument is in the form.

$e \to \neg p \vee \neg c$
$\neg e$

$\therefore \quad p \wedge c$

Invalid, inverse error.

d) All students who do not do their homework and do not study the course material will not get a good course grade.
John gets a good course grade.
John did his homework or studied the course material.
Answer:
x consists of all students. $H(x)$: x does the homework. $S(x)$, x studies. $G(x)$, x gets a good grade. Argument is in the form.

$\forall x((\neg H(x) \wedge \neg S(x)) \to \neg G(x))$
$G(John)$

$\therefore \quad H(John) \vee S(John)$

Valid, universal instantiation, contraposition, modus ponens, De Morgan's law, and double negation.

Question 2

Use rules of inference to show that

a)

$\forall x(R(x) \to (S(x) \vee Q(x)))$
$\exists x(\neg S(x))$

$\therefore \quad \exists x(R(x) \to Q(x))$

Answer:

Step		**Reason**
1.	$\forall x(R(x) \to (S(x) \vee Q(x)))$	Premise
2.	$\exists x(\neg S(x))$	Premise
3.	$R(a) \to (S(a) \vee Q(a))$	Universal instantiation from 1
4.	$\neg R(a) \vee (S(a) \vee Q(a))$	Logical equivalence from 3
5.	$(S(a) \vee Q(a)) \vee \neg R(a)$	Commutative law using 4
6.	$S(a) \vee (Q(a)) \vee \neg R(a)$	Associative law using 5
7.	$\neg S(a)$	Existential instantiation from 2
8.	$Q(a) \vee \neg R(a)$	Disjunctive syllogism using 6 and 7
9.	$\neg R(a) \vee Q(a)$	Commutative law using 8
10.	$R(a) \to Q(a)$	Logical equivalence from 9
11.	$\exists x(R(x) \to Q(x))$	Existential generalization from 10

b)

$\forall x(P(x) \vee Q(x)))$
$\forall x((\neg P(x) \wedge Q(x)) \to R(x))$

$\therefore \quad \forall x(\neg R(x) \to P(x))$

Answer:

Step		**Reason**
1.	$\forall x(P(x) \vee Q(x)))$	Premise
2.	$\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$	Premise
3.	$P(a) \vee Q(a)$	Universal instantiation from 1
4.	$(\neg P(a) \wedge Q(a)) \rightarrow R(a)$	Universal instantiation from 2
5.	$\neg(\neg P(a) \wedge Q(a)) \vee R(a)$	Logical equivalence from 4
6.	$(P(a) \vee \neg Q(a)) \vee R(a)$	De Morgan's law using 5
7.	$((P(a) \vee \neg Q(a)) \vee R(a)) \wedge (P(a) \vee Q(a))$	Conjunction using 6 and 3
8.	$(((P(a) \vee \neg Q(a)) \vee R(a)) \wedge P(a))$ $\vee(((P(a) \vee \neg Q(a)) \vee R(a)) \wedge Q(a))$	Distributive law using 7
9.	$(P(a) \wedge ((P(a) \vee \neg Q(a)) \vee R(a)))$ $\vee(Q(a) \wedge ((\neg Q(a) \vee P(a)) \vee R(a)))$	Commutative law using 8
10.	$(P(a) \wedge (P(a) \vee (\neg Q(a) \vee R(a))))$ $\vee(Q(a) \wedge (\neg Q(a) \vee (P(a) \vee R(a))))$	Associative law using 9
11.	$P(a) \vee (Q(a) \wedge (\neg Q(a) \vee (P(a) \vee R(a))))$	Absorption law using 10
12.	$P(a) \vee ((Q(a) \wedge \neg Q(a)) \vee (Q(a) \wedge (P(a) \vee R(a))))$	Distributive law using 11
13.	$P(a) \vee (F \vee (Q(a) \wedge (P(a) \vee R(a)))$	Negation law using 12
14.	$P(a) \vee (Q(a) \wedge (P(a) \vee R(a)))$	Identity law using 13.
15.	$(P(a) \vee Q(a)) \wedge (P(a) \vee (P(a) \vee R(a)))$	Distributive law using 14
16.	$P(a) \vee (P(a) \vee (R(a))$	Simplification from 15
17.	$(P(a) \vee P(a)) \vee R(a)$	Commutative law using 16
18.	$P(a) \vee R(a)$	Idempotent law using 17
19.	$R(a) \vee P(a)$	Commutative law using 18
20.	$\neg(\neg R(a) \vee P(a))$	Double negation law using 19
21.	$\neg R(a) \rightarrow P(a)$	Logical equivalence using 20.
22.	$\forall x(\neg R(x) \rightarrow P(x))$	Universal generalization using 21.

Question 3

Prove that the product of two odd numbers is odd, using

a) a direct proof;

b) an indirect proof;

c) a proof by contradiction.

Answer:
Let x and y be two odd numbers. We want to show xy is an odd number.

a) Direct proof
By definition of odd number, $x = 2a + 1$ and $y = 2b + 1$ for some integers a and b. Then the product $xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$. We know that $2ab + a + b$ is an integer. Thus, $xy = 2(2ab + a + b) + 1$ is odd.

b) Indirect proof
Contraposition: If xy is the product of x and y, and xy is even; then x or y is even.
Assume $x = 2a + c$ and $y = 2b + d$, a and b are integers, c and d are either 0 or 1. Then $xy = 4ab + 2ad + 2bc + cd = 2(2ab + ad + bc) + cd$. Since xy is even, it has the form $xy = 2e$ for some e. Thus, $cd = 0$. Therefore, x or y is even.

c) By contradiction
Assume x and y are odd, but xy is even.
By definition of odd $x = 2a + 1$ $y = 2b + 1$ for some a and b. Then, $xy = 2(2ab + a + b) + 1$ has the form of $2c + 1$, which is odd. But it contradicts to the assumption. Thus, if x and y are odd, then xy is odd.

Question 4

For any integer n, prove that

a) 3 divides $n^3 - n$
Answer:
$n^3 - n = n(n^2 - 1) = n(n+1)(n-1)$ Since n is an integer, then $n - 1$, n, and $n + 1$ are three consecutive integers. So, one of them is divisible by 3. Thus, $n^3 - n$ is divisible by 3.

b) 3 divides one of the integers n, $n + 1$, or $2n + 1$
Answer:
If n or $n + 1$ is divisible by 3, then the statement is true.
Otherwise, $n - 1$ and $n + 2$ are divisible by 3. Thus, the sum of $n - 1$ and $n + 2$ is $2n + 1$ which is also divisible by 3.

c) 3 divides one of n, $n + 2$, or $n + 4$.
Answer:
If $n \equiv 0 \mod 3$, then n is divisible by 3.
If $n \equiv 1 \mod 3$, then $n - 1$ and $n - 1 + 3 = n + 2$ is divisible by 3.
If $n \equiv 2 \mod 3$, then $n - 2$ and $n - 2 + 6 = n + 4$ is divisible by 3.
Therefore, the statement is true.

d) 3 divides one of n, $2n - 1$, or $2n + 1$.
If $n \equiv 0 \mod 3$, then n is divisible by 3.
If $n \equiv 1 \mod 3$, then $n-1$ and $n+2$ are divisible by 3. Then $n-1+n+2 = 2n + 1$ is divisible by 3.
If $n \equiv 2 \mod 3$, then $n-2$ and $n+1$ are divisible by 3. Then $n-2+n+1 = 2n - 1$ is divisible by 3.

Question 5

a) Prove that $\lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$ for all integers n. **Answer:**
If n is even, $n = 2a$ for some a. Then $LHS = \lfloor 2a/2 \rfloor \lceil 2a/2 \rceil = a^2$,
$RHS = \lfloor 4a^2/4 \rfloor = a^2$,
and $LHS = RHS$.
If n is odd, $n = 2a+1$ for some a. Then $LHS = \lfloor (2a+1)/2 \rfloor \lceil (2a+1)/2 \rceil = a(a+1) = a^2 + a$,
$RHS = \lfloor (2a+1)^2/4 \rfloor = \lfloor (4a^2 + 4a + 1)/4 \rfloor = a^2 + a = LHS$.

b) Show that for any integer n, $5|(n^5 - n)$.
Answer:
$n^5 - n = n(n^4 - 1) = n(n-1)(n+1)(n^2+1)$
If $n = 5a$, then $n^5 - n = 5a(n-1)(n+1)(n^2+1)$, which is divisible by 5.
If $n = 5a+1$, then $n-1 = 5a$ and $n^5 - n = n5a(n+1)(n^2+1)$, which is divisible by 5.
If $n = 5a+2$, then $n-2 = 5a$, $n^2+1 = (n-2)(n+2)+5 = 5(a(n+2)+1)$, and $n^5 - n = n(n-1)(n+1)5(a(n+2)+1)$, which is divisible by 5.
If $n = 5a+3$, then $n-3 = 5a$, $n+2 = 5(a+1)$, $n^2+1 = (n-2)(n+2)+5 = 5((n-2)(a+1)+1)$, and $n^5 - n = n(n-1)(n+1)5((n-2)(a+1)+1)$, which is divisible by 5.
If $n = 5a+4$, then $n-4 = 5a$, $n+1 = 5(a+1)$, and $n^5 - n = n(n-1)5(a+1)(n^2+1)$, which is divisible by 5.
Thus, the statement is true for any integer.

c) Show that if n is odd integer, then $n^3 - n$ is a multiple of 24.
Answer:
Since n is odd, $n = 2a+1$ for some a.
$n^3 - n = n(n-1)(n+1) = 4a(a+1)(2a+1)$. Thus, the original statement is equivalence to $a(a+1)(2a+1)$ is divisible by 6.
If $a = 6b$, then it is done.
If $a = 6b+1$, $a(a+1)(2a+1) = a2(3b+1)3(4b+1)$ is divisible by 6.
If $a = 6b+2$, $a(a+1)(2a+1) = 2(3b+1)3(2b+1)(2a+1)$ is divisible by 6.
If $a = 6b+3$, $a(a+1)(2a+1) = 3(2b+1)2(3b+2)(2a+1)$ is divisible by 6.
If $a = 6b+4$, $a(a+1)(2a+1) = 2(3b+2)(a+1)3(4b+3)$ is divisible by 6.
If $a = 6b+5$, $a(a+1)(2a+1) = a6(b+1)(2a+1)$ is divisible by 6.
Thus, $a(a+1)(2a+1)$ is divisible by 6 for any a.
Thus, the original statement is true for any n.

Question 6

For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, e.g., by giving a counterexample.

a) For all positive $x, y \in \mathbb{Q}$, if x is irrational and y is irrational then $x + y$ is irrational.
Answer: False, counterexample: $x = \sqrt{2}$ and $y = 2 - \sqrt{2}$.

b) $\forall x, y \in \mathbb{Q}$, if x is irrational and y is rational then $x - y$ is irrational.
Answer: True.
Assume x is irrational and y is rational, but $x - y$ is rational. $y = a/b$ and $x - y = c/d$. Then $x = y + (x - y) = a/b + c/d = (ad + bc)/bd$, which is rational. But, x is irrational by assumption. This is a contradiction.

c) $\sqrt{3}$ is irrational.
Answer: True.
Assume $\sqrt{3}$ is rational. Let $\sqrt{3} = a/b$. Then $3 = a^2/b^2$. Thus, $3b^2 = a^2$. a^2 is divisible by 3. a is divisible by 3. So, let $a = 3c$. By substitution, $3b^2 = 9c^2$. Then, $b^3 = 3c^2$. This implies b is also divisible by 3. So, a, b have the common divisor 3, which is a contradiction.

d) $\log_5(2)$ is irrational.
Answer: True
Assume $\log_5(2)$ is rational. $\log_5(2) = a/b$. Hence,

$$\begin{aligned} b\log_5(2) &= a \\ \log_5(2^b) &= a \\ 2^b &= 5^a \\ 5^a &= 2 * 2^{b-1} \end{aligned}$$

So, 5^a is even and divisible by 2. But this is impossible, because 5^a is only divisible by 5. Therefore, the assumption is false.

Question 7

a) Show that if x and y are positive rational numbers with $x < y$ then there is a rational number r with $x < r < y$.
Answer:
Let $a = \frac{y-x}{2}$. Since $x < y$, $\frac{y-x}{2} > 0$. Then, consider the number $r = x + a$. $r - x = a > 0$ and $y - r = a > 0$. Thus, $x < r < y$.

b) Prove that if x is an integer then $\lceil x \rceil = \lfloor x \rfloor$.
Answer:
By definition of ceiling function $\lceil x \rceil = x$ for integer x. And by definition of floor function $\lfloor x \rfloor = x$ for integer x. Thus, $\lceil x \rceil = \lfloor x \rfloor$ is true for any integer x.

c) Prove that if x is a real number but not an integer then $\lceil x \rceil - \lfloor x \rfloor = 1$.
Answer:
Since x is a real number but not an integer, $0 < \lceil x \rceil - x < 1$ and $0 < x - \lfloor x \rfloor < 1$. So, $\lceil x \rceil - \lfloor x \rfloor \in (0, 2)$. We know that $\lceil x \rceil$ and $\lfloor x \rfloor$ are integers. So, $\lceil x \rceil - \lfloor x \rfloor$ is also an integer. And there is only one integer in $(0, 2)$, which is 1. Thus, $\lceil x \rceil - \lfloor x \rfloor = 1$.

Question 8

a) Prove that for any integer n, $n(n^2 - 1)(n + 2)$ is divisible by 12.
Answer:
$n(n^2 - 1)(n + 2) = n(n - 1)(n + 1)(n + 2)$
Since, $n - 1$, n, $n + 1$, and $n + 2$ are 4 consecutive integers, exactly one of them has to be divisible by 4, and at least one of them is divisible by 3. Thus, the product is divisible by 12.

b) Prove that $n \in \mathbb{Z}^+$ is divisible by 3 if and only if the sum of its digits is divisible by 3.
Answer:
Let $n = d_1 + 10d_2 + \cdots + 10^i d_i$
$\Rightarrow$ By contradiction, assuming n is divisible by 3, but $d_1 + d_2 + \cdots + d_i$ is not divisible by 3. Thus,

$$
\begin{aligned}
d_1 + d_2 + \cdots + d_i &\equiv 1 \text{ or } 2 \mod 3 \\
d_2 + \cdots + d_i &\equiv -d_1 + 1 \text{ or } -d_1 + 2 \mod 3 \\
10(d_2 + \cdots + d_i) &\equiv -d_1 - 9d_1 + 10 \text{ or } -d_1 - 9d_1 + 20 \mod 3 \\
10d_2 + \cdots + 10d_i &\equiv -d_1 + 1 \text{ or } -d_1 + 2 \mod 3 \\
10d_3 + \cdots + 10d_i &\equiv -10d_2 - d_1 + 1 \text{ or } -10d_2 - d_1 + 2 \mod 3 \\
10(10d_3 + \cdots + 10d_i) &\equiv -10 * 10d_2 - 10d_1 + 10 \text{ or } -10 * 10d_2 - 10d_1 + 20 \mod 3 \\
100d_3 + \cdots + 100d_i &\equiv -10d_2 - 90d_2 - d_1 - 9d_1 + 1 + 9 \text{ or } -10d_2 - 90d_2 - d_1 - 9d_1 + 2 + 18 \mod 3 \\
100d_3 + \cdots + 100d_i &\equiv -10d_2 - d_1 + 1 \text{ or } -10d_2 - d_1 + 2 \mod 3 \\
&\vdots \\
0 &\equiv -(10^i + \cdots + 10d_2 + d_1) + 1 \text{ or } -(10^i + \cdots + 10d_2 + d_1) + 2 \mod 3 \\
0 &\equiv -n + 1 \text{ or } -n + 2 \mod 3 \\
n &\equiv 1 \text{ or } 2 \mod 3
\end{aligned}
$$

This contradicts to n is divisible by 3.
$\Leftarrow$
$d_1 + d_2 + \cdots d_i \equiv 0 \mod 3$, then same as above.

c) $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$

Answer:

If x is an integer, $LHS = 3x$, $RHS = x + x + x = 3x$, and $LHS = RHS$.
If x is not an integer, let $a = \lfloor x \rfloor$ and $b = \lfloor x \rfloor - a$. Then we have three cases for b.
If $0 < b < \frac{1}{3}$, then $3b < 1$, $b + \frac{2}{3} < 1$, $LHS = \lfloor 3a + 3b \rfloor = 3a$, $RHS = \lfloor a + b \rfloor + \lfloor a + b + \frac{1}{3} \rfloor + \lfloor a + b + \frac{2}{3} \rfloor = a + a + a = 3a$, and $LHS = RHS$.
If $\frac{1}{3} \leq b < \frac{2}{3}$, then $1 \leq 3b < 2$, $b + \frac{1}{3} < 1$, $1 \leq b + \frac{2}{3} < 2$, $LHS = \lfloor 3a + 3b \rfloor = 3a + 1$, $RHS = \lfloor a + b \rfloor + \lfloor a + b + \frac{1}{3} \rfloor + \lfloor a + b + \frac{2}{3} \rfloor = a + a + a + 1 = 3a + 1$, and $LHS = RHS$.
If $\frac{2}{3} \leq b < 1$, then $2 \leq 3b < 3$, $1 \leq b + \frac{1}{3} < 2$, $2 \leq b + \frac{2}{3} < 2$, $LHS = \lfloor 3a + 3b \rfloor = 3a + 2$, $RHS = \lfloor a + b \rfloor + \lfloor a + b + \frac{1}{3} \rfloor + \lfloor a + b + \frac{2}{3} \rfloor = a + a + 1 + a + 1 = 3a + 2$, and $LHS = RHS$.
Thus, $LHS = RHS$ for any x.