Question	1-5	6-10	11	12	13	14	Total
Max	5	5	3	2	3	2	20
Marks							

[5pts]

Multiple-choice: please write your selection in the boxes provided here. You do not need to show any work. A correct answer is worth 1 point; an incorrect answer is worth 0 points.

Question	1	2	3	4	5
Answer	D	C	C	C	A

1. Let $f(x) = 3x\sqrt{\log(x-1)}$. What is the range of f^{-1}?

   A. $\{y \in \mathbb{R} : y > 1\}$
   B. $\mathbb{R}$
   C. $\{y \in \mathbb{R} : y \geq 1\}$
   D. $\{y \in \mathbb{R} : y \geq 2\}$
   E. $\{y \in \mathbb{R} : y \geq 0\}$

Range of f^{-1} = domain of f

We need

$\log(x-1) \geq 0$ and $x-1>0$

$\Downarrow$ $\Downarrow$

$x-1 \geq 10^0 = 1$ $x>1$

$\Rightarrow x \geq 2$

Since $x \geq 2$ is more restrictive, $x>1$ is redundant.

2. Let $f(x) = \ln x$ and $g(x) = \sqrt[3]{x^2}$. What is the domain of $f \circ g$?

   A. $\{x \in \mathbb{R} : x \geq 0\}$
   B. $\mathbb{R}$
   C. $\{x \in \mathbb{R} : x \neq 0\}$
   D. $\{x \in \mathbb{R} : x > 0\}$
   E. $\{x \in \mathbb{R} : x \geq 1\}$

$(f \circ g)(x) = \ln(\sqrt[3]{x^2})$

$g(x)$ has no domain restrictions, so we just need $\sqrt[3]{x^2} > 0$

$\Rightarrow x^2 > 0$

$\Rightarrow |x| > 0$

So any x except $x = 0$ is fine.

3. Let $f(x) = \frac{2+x}{1-x}$. What is $f^{-1}(x)$? (Keep in mind that you may need to simplify your answer.)

   A. $f^{-1}(x) = \frac{x-2}{x-1}$
   B. $f^{-1}(x) = \frac{x+2}{x+1}$
   C. $f^{-1}(x) = \frac{x-2}{x+1}$
   D. $f^{-1}(x) = \frac{2-x}{x+1}$
   E. $f^{-1}(x) = \frac{x-2}{1-x}$

$y = \frac{2+x}{1-x}$

Swap $x \Leftrightarrow y$:

$x = \frac{2+y}{1-y}$

$\Rightarrow x(1-y) = 2+y$

$\Rightarrow -yx - y = 2 - x$

$\Rightarrow y(-x-1) = 2-x$

$\Rightarrow \quad y = \frac{2-x}{-x-1} \cdot \frac{-1}{-1}$

$= \frac{x-2}{x+1}$

4. Let $f(x) = \begin{cases} \sqrt{x} & : \ x > 1 \\ e^x & : \ x \leq 1. \end{cases}$

For what values of x is $f(x)$ **not** differentiable?

A. $\{x \in \mathbb{R} : x < 0\}$

B. f is not differentiable anywhere.

C. $x = 1$

D. f is differentiable everywhere.

E. $\{x \in \mathbb{R} : x < 0 \text{ or } x = 1\}$

First, we know $\sqrt{x}$ and e^x are cts and differentiable everywhere. We need only check what happens at $x=1$

Check if f is cts at $x=1$: $\sqrt{1} \neq e^1$, so f isn't even cts at $x=1$. So f can't be differentiable either.

(Note: even if we missed that step, we would check that

$$\lim_{x\to 1^-} f'(x) \stackrel{?}{=} \lim_{x\to 1^+} f(x)$$

$$\frac{1}{2\sqrt{1}} \neq e^1$$

So f still wouldn't be differentiable even if we forgot to check continuity.)

5. What is the value of $\lim_{x\to\infty} \frac{3x^2-5x+2}{2x^3+4x}$?

A. 0

B. ∞

C. $\frac{3}{2}$

D. $-\infty$

E. The limit does not exist and is neither ∞ nor $-\infty$.

$$\lim_{x\to\infty} \frac{3x^2-5x+2}{2x^3+4x} = \lim_{x\to\infty} \frac{3/x - 5/x^2 + 2/x^3}{2 + 4/x^2}$$

$$= \frac{0}{2} = 0$$

[5pts] **Differentiate the following.**

Answer in the space provided. You do not need to show any intermediate steps or simplify your answers.

6. $f(x) = \frac{1-x^2}{1+e^x}$

$f'(x) = \dfrac{-2x(1+e^x) - (1-x^2)e^x}{(1+e^x)^2}$ (Quotient rule)

7. $f(x) = \underbrace{2^{\sin x}} \cdot \underbrace{3^x}$ (Product rule first)

$f'(x) = 2^{\sin x} \ln 2 \cdot \cos x \cdot 3^x + 2^{\sin x} \cdot 3^x \ln 3$

8. $f(x) = \tan^2(x^2 - 4) = (\tan(x^2-4))^2$

$f'(x) = 2\tan(x^2-4) \cdot \sec^2(x^2-4) \cdot 2x$ (Chain rule x2)

9. $f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$

$f'(x) = -\frac{1}{2}(1+x)^{-3/2}$ or $\dfrac{-1}{2\sqrt{(1+x)^3}}$

10. $f(x) = \underbrace{(x+1)(x+2)}\underbrace{\cos(x+3)}$ There are a couple different ways to break this down.

$f'(x) = [1(x+2) + (x+1) \cdot 1]\cos(x+3) + (x+1)(x+2) \cdot -\sin(x+3)$

Long-answer questions: in questions 11-14, you must show all relevant steps. A correct answer without any justification will not receive full marks.

[3pts]

11. Find the value of

$$\lim_{x\to 1} \frac{x^2-4x+5}{x-1}.$$ ← does NOT factor!

If the limit does not exist, determine whether it is $\infty, -\infty$, or neither.

If we let $x=1$, we get $\frac{1^2-4\cdot 1+5}{1-1} = \frac{2}{0}$, so for sure the limit DNE. We must check one-sided limits to try to classify it.

$$\lim_{x\to 1^-} \frac{x^2-4x+5}{x-1} = \frac{2}{\ominus} = -\infty$$

$$\lim_{x\to 1^+} \frac{x^2-4x+5}{x-1} = \frac{2}{\oplus} = +\infty$$

not equal, so limit is neither ∞ nor $-\infty$

[2pts]

12. Find all values $c \in \mathbb{R}$ such that the function

$$f(x) = \begin{cases} (x-1)^2 & : x \geq 1 \\ cx^2 + cx - 2 & : x < 1 \end{cases}$$

is **continuous**.

Both pieces are polynomials, so f is cts everywhere except maybe at $x=1$.

We need $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$$\underset{c\cdot 1^2 + c\cdot 1 - 2}{\parallel} \qquad \underset{(1-1)^2}{\parallel} \qquad \underset{(1-1)^2}{\parallel}$$

So we need $2c-2=0$

$\Rightarrow c=1$.

$\therefore$ f is cts when $c=1$.

[3pts]

13. Find the derivative of $f(x) = \sqrt{x^2 - 2x - 1}$ by using the definition of the derivative as a limit. *You* **must** *use the definition of the derivative. A correct answer obtained by any other method will not receive any points.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

note: don't expand this right away! It will just make everything longer.

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 2(x+h) - 1} - \sqrt{x^2-2x-1}}{h} \cdot \frac{\sqrt{(x+h)^2 - 2(x+h) - 1} + \sqrt{x^2-2x-1}}{\sqrt{(x+h)^2 - 2(x+h) - 1} + \sqrt{x^2-2x-1}}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - 1 - (x^2 - 2x - 1)}{h\left(\sqrt{(x+h)^2 - 2(x+h) - 1} + \sqrt{x^2-2x-1}\right)}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - 1 - x^2 + 2x + 1}{h\left(\sqrt{(x+h)^2 - 2(x+h) - 1} + \sqrt{x^2-2x-1}\right)}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}} - 2\cancel{h}}{\cancel{h}\left(\sqrt{(x+h)^2 - 2(x+h) - 1} + \sqrt{x^2-2x-1}\right)}$$

$$= \lim_{h \to 0} \frac{2x + h - 2}{\sqrt{(x+h)^2 - 2(x+h) - 1} + \sqrt{x^2-2x-1}}$$

$$= \frac{2x - 2}{2\sqrt{x^2 - 2x - 1}}$$

[2pts] 14. Find the equation of the line tangent to $y = \sin(e^{x-1} - 1)$ at $x = 1$.

We need to find m and b for $y=mx+b$.

Find m: $\frac{dy}{dx} = \cos(e^{x-1}-1)\cdot e^{x-1}$

$m = \left.\frac{dy}{dx}\right|_{x=1} = \cos(e^{0}-1)\cdot e^{0}$

$= \cos(0)\cdot 1$

$= 1$

Find a point (x,y) on the tangent line:

$y\big|_{x=1} = \sin(e^{1-1}-1)$

$= \sin(0) = 0$

So $(1,0)$ is a point.

Sub $x=1, y=0, m=1$ into $y=mx+b$:

$0 = 1\cdot 1 + b$

$\Rightarrow b = -1$.

$\therefore$ the equation of the tangent line is

$y = x - 1$.