Problem 5

$x^2 y'' + \lambda y = 0 \qquad y(1) = 0 = y(2)$

let $y = x^r$, $y' = r x^{r-1}$ $\quad y'' = r(r-1)x^{r-2}$

$r(r-1)x^{r-2}x^2 + \lambda \cdot x^r = 0$

$$r(r-1) + \lambda = 0$$
$$r^2 - r + \lambda = 0$$

Only consider $\lambda > 0$ case, $\lambda = \mu^2$

$$r^2 - r + \mu^2 = 0$$

$\Delta = 1 - 4\mu^2$

$r = \dfrac{1 \pm \sqrt{1-4\mu^2}}{2}$

① $\mu < \frac{1}{2} \quad \sqrt{1-4\mu^2} > 0$

$r_{1,2} = \dfrac{1 \pm \sqrt{1-4\mu^2}}{2}$

$y = C_1 x^{\frac{1+\sqrt{1-4\mu^2}}{2}} + C_2 x^{\frac{1-\sqrt{1-4\mu^2}}{2}}$

$\because y(1) = 0 = y(2)$

$\therefore 0 = C_1 \cdot 1 + C_2 \cdot 1$

$\Rightarrow C_1 + C_2 = 0$

$0 = C_1 \cdot 2^{\frac{1+\sqrt{1-4\mu^2}}{2}} + C_2 \cdot 2^{\frac{1-\sqrt{1-4\mu^2}}{2}}$

$0 = C_2 \left(-2^{\frac{1+\sqrt{1-4\mu^2}}{2}} + 2^{\frac{1-\sqrt{1-4\mu^2}}{2}}\right)$

$\Rightarrow C_2 = 0 \quad \Rightarrow C_1 = 0$

Discard this case $\mu < \frac{1}{2}$, since it produces a trivial solution

② $\mu = \frac{1}{2}$

$r = \dfrac{1 \pm \sqrt{1-4\mu^2}}{2}$

$r = \frac{1}{2}$

$y = C_1 x^{\frac{1}{2}} + C_2 x^{\frac{1}{2}} \ln|x|$

$y(1) = 0 = y(2)$

$0 = C_1 \cdot 1 + C_2 \cdot 1 \cdot 0 \quad \Rightarrow C_1 = 0$

$y = C_2 x^{\frac{1}{2}} \ln|x|$

$0 = C_2 \cdot 2^{\frac{1}{2}} \ln 2 \quad \Rightarrow C_2 = 0$

Discard this case $\mu = \frac{1}{2}$, since it produces a trivial solution

③ $\mu > \frac{1}{2}$

$$r_{1,2} = \frac{1 \pm \sqrt{1-4\mu^2}}{2}$$

$$= \frac{1 \pm \sqrt{4\mu^2-1}\, i}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{4\mu^2-1}}{2} i$$

$$y = x^{\frac{1}{2}} \left[C_1 \cos\left(\frac{\sqrt{4\mu^2-1}}{2} \ln|x|\right) + C_2 \sin\left(\frac{\sqrt{4\mu^2-1}}{2} \ln|x|\right) \right]$$

$\because$ $y(1) = 0 = y(2)$

$$0 = C_1 \cos(0) + C_2 \sin(0)$$

$$0 = C_1 \cdot 1 + C_2 \cdot 0 \Rightarrow C_1 = 0$$

$$y = x^{\frac{1}{2}} \cdot C_2 \sin\left(\frac{\sqrt{4\mu^2-1}}{2} \ln|x|\right)$$

$$0 = 2^{\frac{1}{2}} C_2 \sin\left(\frac{\sqrt{4\mu^2-1}}{2} \ln 2\right)$$

$$\sin\left(\frac{\sqrt{4\mu^2-1}}{2} \ln 2\right) = 0$$

$$\frac{\sqrt{4\mu^2-1}}{2} \ln 2 = n\pi$$

$$\frac{\sqrt{4\mu^2-1}}{2} = \frac{n\pi}{\ln 2}$$

$$\sqrt{4\mu^2-1} = \frac{2n\pi}{\ln 2}$$

$$4\mu^2 - 1 = \left(\frac{2n\pi}{\ln 2}\right)^2$$

$$\mu^2 = \frac{\left(\frac{2n\pi}{\ln 2}\right)^2 + 1}{4}$$

$$\mu = \sqrt{\frac{\left(\frac{2n\pi}{\ln 2}\right)^2 + 1}{4}}$$

$$y = C_2 x^{\frac{1}{2}} \sin\left(\frac{\sqrt{4\mu^2-1}}{2} \ln x\right)$$

where $\mu = \sqrt{\frac{\left(\frac{2n\pi}{\ln 2}\right)^2 + 1}{4}}$

Problem 6

let $U = X_{(x)}T(t)$ $\quad U_t = \alpha^2 U_{xx}$

$U_t = 0.2\, U_{xx}$

$xT' = 0.2\, x''T$

$\frac{x''}{x} = \frac{T'}{0.2T} = \lambda$

x problem $\frac{x''}{x} = \lambda$ $\quad x'' - \lambda x = 0$ $\quad r^2 - \lambda = 0$

① $\lambda > 0$, $\lambda = \mu^2$

$r^2 - \mu^2 = 0$

$r^2 = \mu^2$

$r = \pm\mu$

$\therefore X = A\cosh(\mu x) + B\sinh(\mu x)$

Apply BC : $U_x(0,t) = 0 = u(1,t)$

$x' = \mu A\sinh(\mu x) + \mu B\cosh(\mu x)$

$x'(0) = \mu A\sinh(\mu 0) + \mu B\cosh(\mu \cdot 0) = 0$

$\mu \cdot A \cdot 0 + \mu \cdot B \cdot 1 = 0 \Rightarrow \underline{B = 0}$

$X = A\cosh(\mu x)$

$X(1) = 0 \Rightarrow x(1) = A\cosh(\mu \cdot 1) \Rightarrow \underline{A = 0}$

② $\lambda = 0$ $\quad x'' - \lambda x = 0 \Rightarrow x'' = 0$

$x' = \int 0 dx = A$

$x = \int A\,dx = Ax + B$

$x = Ax + B$

Apply BC $\quad U_x(0,t) = 0 = u(1,t)$

$x' = A \begin{cases} u_x(0,t) = 0 \\ x'(0) = 0 \end{cases}$

$\underline{A = 0}$

$u(1,t) = 0 \Rightarrow \underline{B = 0}$

③ $\lambda < 0$ $\quad \lambda = -\mu^2$

$X'' - \lambda x = 0$ $\quad r^2 - \lambda = 0$

$r^2 + \mu^2 = 0$

$r^2 = -\mu^2$

$r = \pm\mu i$ $\quad$ complex eigenvalue.

$\therefore X = A\cos(\mu x) + B\sin(\mu x)$

Apply BC $U_x(0,t)=0=U(1,t)$

$x' = -\mu A\sin(\mu x)+\mu B\cos(\mu x)$

$x'(0) = -\mu\cdot A\cdot 0+\mu\cdot B\cdot 1=0 \Rightarrow B=0$

$X = A\cos(\mu x)$

$X(1) = A\cos(\mu)=0$

$\cos\mu = 0$

$\mu = \frac{(2n+1)\pi}{2} \quad n\in 0.1.2.3.4$

$X_n = A_n\cos\left(\frac{(2n+1)\pi}{2}x\right)$

T problem : $\frac{T'}{0.2T} = \lambda$

$\lambda = -\mu^2 \cdot \mu = \frac{(2n+1)\pi}{2}$

$\frac{T'}{0.2T} = -\mu^2$

$T' = -0.2\mu^2 T$

$T = De^{-0.2\mu^2 T}$

$y' = ky$

$\frac{dy}{dt} = ky$

$\int\frac{dy}{y} = \int k\cdot dt$

$\ln y = kt+C$

$y = e^{kt+C} = De^{kt}$

$u(x,t) = X(x)\cdot T(t)$

$\therefore u(x,t) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{(2n+1)\pi}{2}x\right)e^{-0.2\left(\frac{(2n+1)\pi}{2}\right)^2 t}$

Apply IC $u(x,0) : \cos\left(\frac{3}{2}\pi x\right)$

$u(x,0) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{(2n+1)\pi x}{2}\right) = \cos\left(\frac{3\pi x}{2}\right)$

$u(x,0) = A_0\cos\left(\frac{1}{2}\pi x\right)+A_1\cos\left(\frac{3\pi}{2}x\right)+A_2\cos\left(\frac{5\pi}{2}x\right)+\cdots$

$= \cos\left(\frac{3\pi x}{2}\right)$

We take $n=1$ $A_1=1$

$A_n = \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases}$

$u(x,t) = A_1\cos\left(\frac{(2\cdot 1+1)\pi}{2}x\right)e^{-0.2\left(\frac{(2\cdot 1+1)\pi}{2}\right)^2 t}$

$= \cos\left(\frac{3}{2}\pi\right)xe^{-0.2\left(\frac{3\pi}{2}\right)^2 t}$

$u(x,0) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{(2n+1)\pi x}{2}\right) = f(x)$

$A_n = \frac{2}{L}\int_0^L f(x)\cdot X_n \text{ eig } dx$

$A_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{(2n+1)\pi}{2}x\right)dx$

$L: 0<x<L$ $f(x): I.C$

$f(x+\Delta x) = f(x)+\Delta x f'(x)+\frac{\Delta x^2}{2!}f''(x)+\frac{\Delta x^3}{3!}f'''(x)+\cdots$

$f(x+\Delta x)-f(x) = \Delta x f'(x)+\frac{\Delta x^2}{2!}f''(x)+\cdots\cdots$

$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)+\frac{\Delta x}{x!}f''(x)+\cdots$

$= f'(x)+O(\Delta x)$

$\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{\Delta x^2} = f''(x)+\frac{1}{12}\Delta x^2 f''''(x)$

$= f''(x)+O(\Delta x^2)$

$$u_t = \alpha^2 u_{xx}$$

$$\frac{u(x, t+\Delta t) - u(x, t)}{\Delta t} = \alpha^2 \left(\frac{u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t)}{\Delta x^2} \right) + O(\Delta t, \Delta x^2)$$

$$u(x, t+\Delta t) - u(x, t) = \alpha^2 \left(\frac{\Delta t}{\Delta x^2}\right) \left(u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t)\right)$$

$$u(x, t+\Delta t) = u(x, t) + \alpha^2 \left(\frac{\Delta t}{\Delta x^2}\right) \left(u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t)\right)$$

$$u(x, t) = u_n^k$$

$$u_n^{k+1} = u_n^k + \alpha^2 \left(\frac{\Delta t}{\Delta x^2}\right)\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$