Practice Problems

MAT1300

§ 1.1.

1. Draw a diagram, as given in lecture notes, to show the sequence of operations used to evaluate each of the following functions:

a. $\left(\dfrac{3x+2}{x^2}\right)^{10}$. b. 2^{x^2-1}.

2. Suppose a function $y = f(x)$ defined on closed interval [0, 10] satisfies the following conditions:

(i) $f(0) = -6, f(2) = 4, f(7) = -4, f(10) = 1$

(ii) The function increase when $0 < x < 2$ or $7 < x < 10$, and it decreases when $2 < x < 7$.

(iii) The graph of this function is concave up when $5 < x < 10$, and concave down when $0 < x < 5$.

Questions:

(a) At which value(s) of x, does this function attain a local maximum?

(b) At which value(s) of x. does this function attain a local minimum?

(c) At which value(s) of x, does this function attain a global maximum?

(d) At which value(s) of x, does this function attain a global minimum?

(e) At which value(s) of x, does the graph of this function have inflection point(s)?

(f) Sketch the graph of this function.

3. If the demand function of a product is given by $p = 4 - 0.01q^2$, $0 \le q \le 20$, and the cost function is given by $C = 0.5q + 100$, where p is the price and q is the quantity. Find the profit function $P(q)$ of this product.

§1.2.

1. The following table gives the some values of two functions $f(x)$ and $g(x)$:

x	1	2	3	4
$f(x)$	3	2	4	1
$g(x)$	2	4	1	3

Fill in the following table:

x	1	2	3	4
$(f \circ g)(x)$				
$(g \circ f)(x)$				
$(f \circ f)(x)$				
$(g \circ g)(x)$				

2. Decompose each of the following functions into the composition of two functions with an intermediate variable u such that y is obtained from u with only one operation:

a. $\left(\dfrac{3x+2}{x^2}\right)^{10}$.

b. 2^{x^2-1}.

§ 1.3.

1. Suppose the equation of a line L is $3x - 2y = 1$. Find the equation of a line L_1 that goes through a point $(4, -3)$ if

a. L and L_1 are parallel;

b. L and L_1 are perpendicular.

2. Suppose the demand function of a product is linear. When the price is \$40 each, 100 items can be sold. When the price is increased to \$42 each, only 92 items can be sold. Find the demand function in two ways:

a. Express the price p as a function of quantity q, or

b. Express the quantity q as a function of the price p.

§ 1.4.

1. Simplify the following expressions:

a. $\dfrac{\sqrt{x}\,y^{-1/2}}{x^{3/2}y^{1/2}}$.

b. $\left(\dfrac{\sqrt{x}\,y^{2}}{z^{1/2}}\right)^{2}$.

c. $\left(\dfrac{1}{\sqrt{x}}+\sqrt{x}\right)x^{3/2}$.

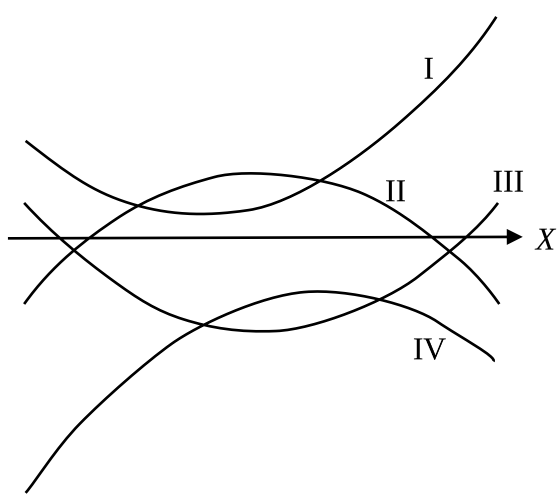
§ 1.5.

1. Factorize the following polynomials:

a. $2x^2 - 5x + 2$,

b. $x^3 - 2x^2 + x$.

2. The following figure shows the graphs of the functions $y_1 = x^2 - x + 1$, $y_2 = x^2 - 2x - 4$, $y_3 = -x^2 + 3x - 4$, $y_4 = -x^2 + 2x + 1$. Match the graphs with these functions.



§ 2.1.

1. Find the following limits:

a. $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-3}$.

b. $\lim\limits_{x\to 3} \dfrac{1}{x-3}\left(\dfrac{1}{2x-1}-\dfrac{1}{x+2}\right)$.

c. $\lim\limits_{x\to 3} \dfrac{2-\sqrt{x+1}}{x-3}$.

2. If function $y = \begin{cases} ax+5, & x<1 \\ 3-bx, 1\le x\le 3 \\ (a-b)x, & x>3 \end{cases}$ is continuous for all values of x, find a and b.

3. Find the vertical and horizontal asymptote(s), if any, of the function $f(x) = \dfrac{x^2-3x+2}{2x^2-x-1}$. If $x = a$ is a vertical asymptote, find $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$.

§ 1.6.

1. For which values of x is each of the following function positive?

a. $y = (x-1)^2(x-3)(x-5)$;

b. $y = \dfrac{(x+3)(x-4)}{x^2(x-2)}$.

§ 1.7.

1. Suppose \$1000 is deposited to a bank account with annual interest rate 3%. Find the balance after five years if

a. the interest is compounded every three months,

b. the interest is compounded every four months,

c. the interest is compounded continuously.

2. Suppose the population of a country was 1.5 million at the beginning of 2014, and the population is increasing at a rate 1% very year. Predict the population of this country at the beginning of 2020.

§ 1.8.

1. Find the following logarithms:

a. $\log_3 1$; b. $\log_3 3$; c. $\log_3 \frac{1}{9}$; d. $\log_3 \sqrt{3}$; e. $\log_3 \frac{1}{\sqrt{3}}$.

2. Simplify:

a. $3^{2\log_3 2}$; b. $\ln e^2$.

3. Solve the following equations:

a. $\log_2 (x + 2) + \log_2(x - 1) = 2$.

b. $2^{x+1} = 3^x$.

4. Suppose the sales of a product decreases exponentially. In January, 8000 items were sold, and in May, 7200 items were sold. What would be the sales of this product in November?

5. If we want the balance in an account to double in 10 years, what should be the annual interest rate, assuming the interest is compounded continuously?

§ 2.2.

1. Consider function $y = \dfrac{x}{2x-3}$.

(a) Find the slope of the secant line that joins two points on the graph of this function where $x = 2$ and $x = 3$.

(b) Find the derivative of this function **by the definition of the derivative**.

(c) Find the equation of the tangent line of the graph of this function at $x = 2$.

(d) Determine for which values of x is this function increasing? For which values of x is this function decreasing?

2. Suppose the position function of a particle moving along the x-axis is given by $x = \dfrac{1}{\sqrt{3t+2}}$. Find the velocity function of this particle by the definition of the derivative.

3. Suppose the profit function of a product is given by $P(q) = 30q - 0.1\sqrt{q}$, where q is the quantity sold. Find the marginal profit of this product at $q = 100$ by the definition of the product.

§ 2.3, 2.4.

1. Find the derivative of the function $y = \dfrac{x^2 + x + 1}{\sqrt{x}}$.

2. Find the equation of a line L that goes through point $(1, -4)$ and is perpendicular to the tangent line of the graph of function $y = \dfrac{x^2+1}{x-1}$ at the point where $x = 3$.

3. For which value(s) of x does the graph of the function $y = \dfrac{\ln x}{\sqrt{x}}$ have a horizontal tangent line?

4. In which interval(s) is the graph of function $y = xe^x$ increasing?

§ 2.5.

1. Find the derivative of the following functions:

a. $y = \sqrt{\ln x}$.

b. $y = x\,e^{x^2+2x}$.

c. $y = \dfrac{\ln(x^2+1)}{x}$.

§ 2.11.

1. Suppose a function $y = f(x)$ is defined implicitly by the equation $x^2y + y^3 + 5 = 0$. Find the derivative $f'(x)$ at the point $(2, -1)$.

2. Find the equation of the tangent line of the graph of the equation $xy^2 + 3x^2 - 3y = 1$ at the point (1, 2).

§ 2.10.

1. Use the tangent line approximation of the function $y = \dfrac{1}{\sqrt{5x-6}}$ at $x = 3$ to estimate the value of $\dfrac{1}{\sqrt{10}}$.

2. Suppose that the demand function of a certain product is $q = \sqrt{180-p}$, $0 \le p \le 180$. For which values of p is the demand elastic, and for which values of p is the demand inelastic?

§ 2.6.

1. Consider function $y = 3x^5 - 10x^4 + 10x^3$. For which values of x is the graph of this function concave up? For which values of x is the graph of this function concave down? Find all inflection point(s), if any.

2. Consider function $y = x^3 + \dfrac{1}{x^2}$. For which values of x is the graph of this function concave up? For which values of x is the graph of this function concave down? Find all inflection point(s), if any.

3. Consider function $y = \dfrac{\ln(x)}{x}$, $x > 0$. For which values of x is the graph of this function concave up? For which values of x is the graph of this function concave down? Find all inflection point(s), if any.

§ 2.7.

1. Find the local and global extrema of the function $y = x^4 - 6x^3 + 9x^2$, $-1 < x < 2$.

§ 2.8.

1. Suppose that a function $y = f(x)$ defined and continuous for all $x \neq 1$ satisfies the following conditions:

(i) $f'(x) > 0$ when $x < 1$, and $f'(x) < 0$ when $x > 1$;
(ii) $f''(x) < 0$ when $x < 0$; $f''(x) > 0$ when $0 < x < 1$ or $x > 1$;
(iii) $f(0) = f(2) = 0$;
(iv) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$;
(v) $\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to \infty} f(x) = -1$.

Use the first and the second derivative analysis to answer the following questions:

a. For which values of x is this function increasing, and, for which values of x is this function decreasing?

b. Find all local extrema of this function, if any.

c. For which values of x is the graph of this function concave up, and, for which values of x is the graph of this function concave down?

d. Find all inflection points, if any.

e. Determine the vertical and horizontal asymptotes, if any.

f. Sketch the graph of this function.

2. Consider function $y = \dfrac{x}{1+x^2}$.

Use the first and the second derivative analysis to find the critical number(s), intervals where this function is increasing/decreasing, concave up/down, local/global extrema, and inflection points. Find the asymptotes, if any, and sketch the graph of this function.

§ 2.9.

1. Suppose $2x + y = 46$, $x \geq 0$, $y \geq 0$. What is the maximum value of $z = xy + 2x$?

2. Suppose the demand function of a product is given by the formula $p = \frac{100}{1+q}$, where p is the price and $q \geq 0$ is the quantity, and the cost function is $C(q) = 2q + 10$. Find q that maximizes the profit.

§ 3.1.

1. With a calculator, use the left sum, right sum and the trapezoidal rule, with $n = 6$ to estimate the definite integral $\int_1^4 \sqrt{x^2+1}\,dx$. Use four digits after the decimal point in your calculation.

2. If $\int_1^3 f(x)dx = 5, \int_{-1}^3 f(x)dx = 8$, what is $\int_{-1}^1 (2-3f(x))dx$?

§ 3.2, 3.3.

1. Find $\int_1^4 \frac{3x+1}{\sqrt{x}}dx$.

2. Find $\int_1^e \left(e^x - \frac{2}{x}\right)dx$.

3. If $F(x)$ is an antiderivative of the function $f(x) = x - \sqrt{x}$, $F(0) = -1$, find $F(4)$.

4. Find $F(3)$ if $F''(x) = \frac{1}{x^2}$, $F'(1) = 0$, $F(1) = 1$.

§ 3.4.

Find the following definite or indefinite integrals:

1. $\int x\sqrt{2x+1}dx$.

2. $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx$.

3. $\int \frac{x^2}{\sqrt{x^3+1}}dx$.

4. $\int \frac{1}{x\sqrt{\ln x}}dx$.

5. $\int_0^1 \frac{e^x}{e^x+1}dx$.

6. $\int \frac{1}{\sqrt{x}+1}dx$.

7. $\int_0^4 \frac{x^3}{\sqrt{x^2+9}}dx$.

§ 3.5.

Find the following integrals:

1. $\int xe^{-2x}dx$.

2. $\int x^2e^{-x}dx$.

3. $\int_1^e x^2 \ln x dx$.

§ 3.6.

1. Consider functions $f(x) = x^2 - x - 2$ and $g(x) = -x^2 - x + 6$.

a. Find the area of the region under the graph of $y = g(x)$ and above the x-axis.

b. Find the area of the region under the graph of $y = g(x)$ and above the graph of $y = f(x)$.

c. Find the area of the region between the graphs of $y = f(x)$ and $y = g(x)$ in the interval $[-1, 3]$.

d. Find the area of the region between the graphs of $y = f(x)$ and $y = g(x)$ in the interval $[-3, 3]$.

2. Suppose the oil price increases in September by the formula $p = \ln(0.1x + 3)$, (dollars/liter) where $0 \le x \le 30$ is the number of days after September 1, 0:00. Find the average price of the oil in September.

§ 3.7.

1. Find the consumer's and producer's surplus:

a. $S(q) = 3q + 1$, $D(q) = (9 - q)^2$, $0 \le q \le 9$.

b. $D(q) = \sqrt{25 - 3q}$, $0 \le q \le 8$, $S(q) = q + 1$.

§ 3.8.

Use the definition to determine the convergence of each of the following improper integrals. If an improper integral is convergent, find its value.

1. $\int_0^\infty \frac{x}{e^x}\,dx$.

2. $\int_0^1 \frac{1}{1-\sqrt{x}}\,dx$.

3. $\int_1^4 \frac{1}{\sqrt{x}-1}\,dx$.

4. $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$.

5. $\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$.

§ 4.1.

1. Describe the natural domain of the following functions

(a) $f(x, y) = \dfrac{\sqrt{x^2 - y^2}}{x + y - 1}$;

(b) $z = \ln(x + y - 1)$.

2. Find the distance between two points (1, 2, 4) and (−1, −1, −2). Find the coordinates of the midpoint of the line segment joining these two points.

3. Find the center and the radius of the sphere with equation $x^2 + y^2 + z^2 + 4x - 6y = 12$.

4. Sketch the contour diagram of the function $z = x^2 + 2y$, with $z = -2, -1, 0, 1$, and 2.

§ 4.2.

1. Find the first and second derivatives of the following functions:

a. $z = x^2y + xy^2 - 3xy$.

b. $z = x^4 + y^4 - 4xy + 1$.

c. $z = e^{x^2 - y^2}$.

d. $z = \ln(x^2 + y)$.

e. $z = \dfrac{\ln x}{y^2}$.

f. $z = x\sqrt{2x - 3y}$.

2. Note that $\sqrt[3]{4 \times 11 + 9^2} = \sqrt[3]{125} = 5$. Use the linear approximation of the function $z = \sqrt[3]{4x + y^2}$ at (11, 9) to find an approximation of $\sqrt[3]{136} = \sqrt[3]{4 \times 9 + 10^2}$.

§ 4.3.

Find the critical points of the following functions and use the second derivative test to determine the characteristic of each critical point:

1. $z = x^3 + 3x^2 + 7x - 4xy + 2y^2 - 8y$.

2. $z = x^3 + 3xy^2 - 30x - 18y$.

3. $z = xy^2 - y^3 + 2y^2 + x^2 + x$.