MAT1341 Final

Linear combination

↳ Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m \in \mathbb{R}^n$ and $k_1, k_2, \ldots, k_m \in \mathbb{R}$

A linear combination of $\bar{v}_1, \ldots, \bar{v}_m$ is any vector of the form $k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_m\bar{v}_m$

Orthoganality

↳ $\bar{x}, \bar{y} \in \mathbb{R}^n$ are orthogonal (perpindicular) if $\bar{x} \cdot \bar{y} = 0$

Angles between vectors

Cosine formula - $\bar{x} \cdot \bar{y} = \|\bar{x}\| \|\bar{y}\| \cos\Theta$

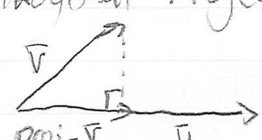
If $\bar{x}, \bar{y} \in \mathbb{R}^n \mid \bar{x}, \bar{y} \neq 0$, we define the angle between $\bar{x}$ and $\bar{y}$

$$\arccos\left(\frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\| \|\bar{y}\|}\right) = \Theta$$

Theorem (Cauchy-Schwartz Inequality)

If $\bar{x}, \bar{y} \in \mathbb{R}^n$ then $|\bar{x} \cdot \bar{y}| \leq \|\bar{x}\| \|\bar{y}\|$

Orthogonal Projections



$$proj_u\bar{v} = \frac{\bar{u} \cdot \bar{v}}{\bar{u} \cdot \bar{u}} \bar{u} = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|^2} \bar{u}$$

$$\|proj_u\bar{v}\| = \left\| \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|^2} \bar{u} \right\| = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|}$$

"Projection of $\bar{v}$ on $\bar{u}$"

System of linear equations

Ex $\begin{cases} x_1 + 2x_2 + x_3 = -1 \\ x_2 - x_3 = 0 \end{cases}$ A solution is a vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ which satisfies all equations

x_3 can be anything, a free variable

say $x_3 = t$, $t \in \mathbb{R}$

Then $x_2 - x_3 = 0$, so $x_2 = x_3 = t$, x_2 is a linked variable

Also $x_1 + 2x_2 + x_3 = -1$, so $x_1 = -1 - 3t$, x_1 is also a linked variable

A linear system is inconsistent if it has no solutions Ex = $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 5 \end{cases}$

If all constant terms are 0, the system is homogeneous

Note $\bar{0}$ is always a solution of any homogeneous system

Any system of linear equations has either:
- a unique solution
- no solution (inconsistent)
- infinate solutions

The <u>rank</u> of a matrix (or system) A is the number of leading ones (linked variables) in the REF of A

Ex $A = \begin{pmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ rank$(A) = 3$

If rank$(A|b) >$ rank(A), inconsistant system (no solutions)
If rank$(A|b) =$ rank(A), consistant system (one or ∞ solutions)

<u>The Span</u>

span$(\bar{v}_1, \ldots, \bar{v}_m)$ = set of all possible linear combinations of $\bar{v}_1, \ldots, \bar{v}_m$

<u>Matrix Multiplication</u>

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 & 9 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot6+2\cdot7 & 1\cdot9+2\cdot8 \\ 3\cdot6+4\cdot7 & 3\cdot9+4\cdot8 \end{pmatrix} = \begin{pmatrix} 20 & 25 \\ 46 & 59 \end{pmatrix}$

In general, if $A = (a_{ij})$ and $B = (b_{ij})$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$

☆ # of columns of A has to be equal to # of rows of B

Then $AB = c_{ij}$ where $c_{ij} = \sum_{l=1}^{n} a_{il} \cdot b_{lj}$

☆ Careful when expanding: $(A+B)^2 = A^2 + BA + AB + B^2$

$A\bar{x} = \bar{b}^{-1}$, $\bar{x} = A^{-1}\bar{b}$

<u>Vector Space</u>: A set of objects which you can add to each other, multiply by scalars, w the 10 rules

A set with 2 operations +, scalar multiplication is a vector space if

1) $\bar{u}, \bar{v} \in V$ then $\bar{u} + \bar{v} \in V$ (closure)
2) $c \in \mathbb{R}$, $\bar{v} \in V$ then $c\bar{v} \in V$
3) V has a zero vector s.t. $\bar{0} + \bar{v} = \bar{v}$ for any $\bar{v} \in V$
4) Every vector $\bar{v} \in V$ has an inverse (negative, $-\bar{v}$) s.t. $\bar{v} + (-\bar{v}) = \bar{0}$
5) $\bar{v}_1 + \bar{v}_2 = \bar{v}_2 + \bar{v}_1$
6) $(\bar{v}_1 + \bar{v}_2) + \bar{v}_3 = \bar{v}_1 + (\bar{v}_2 + \bar{v}_3)$
7) $c(\bar{v}_1 + \bar{v}_2) = c\bar{v}_1 + c\bar{v}_2$
8) $(a+b)\bar{v} = a\bar{v} + b\bar{v}$
9) $1\bar{v} = \bar{v}$
10) $(a \cdot b)\bar{v} = a(b\bar{v})$

$\mathbb{R}^n$

Let W be a vector space, and V be a subset of W such that V is also a vector space with the same operations as W. Then V is called a subspace of W

→ every element of V is also an element of W

Subspace test: If V is a subset of W ($V \subset W$) then it is a subspace
1) $\bar{0} \in V$
2) V is closed under addition
3) V is closed under scalar multiplication

Spans and Linear Combinations

$\bar{v} = c_1\bar{v}_1, c_2\bar{v}_2 + \ldots + c_m\bar{v}_m$ is called a linear combination of $\bar{v}_1, \ldots, \bar{v}_m$

The span of $\bar{v}_1, \ldots, \bar{v}_m$ is the set of all possible linear combinations

so $\text{span}(\bar{v}_1, \ldots \bar{v}_m) = \{c_1\bar{v}_1 + \ldots + c_m\bar{v}_m \mid c_1, \ldots, c_m \in \mathbb{R}\}$

Note: The vectors determining a span are not unique

Theorem: If $\bar{v}_1, \ldots, \bar{v}_m \in V$ is a vector space then $\text{span}(\bar{v}_1, \ldots, \bar{v}_m)$ is always a subspace

Linear dependence (relation)

$\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m$ are said to be linearly dependent if

$c_1\bar{v}_1 + \ldots + c_m\bar{v}_m = \bar{0}$ for some $c_1, \ldots c_m \in \mathbb{R}$ not all zero

Linear independence

$\bar{v}_1, \ldots, \bar{v}_m$ are linearly independent if the only solution to

$c_1\bar{v}_1 + \ldots + c_m\bar{v}_m = \bar{0}$ is $c_1 = c_2 = \ldots = c_m = 0$

Basis and Dimension

The dimension is the least amount of vectors required to span V

The vectors $\bar{v}_1, \ldots, \bar{v}_m$ form a basis of V if
1) $v_1, \ldots, v_m$ are linearly independent
2) $V = \text{span}(\bar{v}_1, \ldots, \bar{v}_m)$

The dimension is the number of vectors in a basis of V

- If $\dim V = n$, then any set of n lin ind vectors forms a basis of V
- Any spanning set $\bar{v}_1, \ldots, \bar{v}_n$ of V consisting of $n = \dim V$ vectors forms a basis

Suppose $\dim V = n$ and W is a subspace of V then
1) $0 \leq \dim W \leq n$
2) $\dim W = \dim V$ IFF $W = V$
3) $\dim W = 0$ IFF $W = \{\bar{0}\}$

Column space (image), row space, and null space (Kernel)

The column space of A is, by definition, $\text{span}(\bar{c}_1, \dots, \bar{c}_n)$ $A = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix}$
and is denoted by $\text{col}(A)$ or $\text{im}(A)$

Row space (less important): $\text{row}(A) = \text{span}(\bar{r}_1, \dots, \bar{r}_m)$ $A = \begin{pmatrix} - & r_1 & - \\ & \vdots & \\ - & r_m & - \end{pmatrix}$

Null space (kernel): "The set of all zeros of the function"
$\text{null}(A) = \text{ker}(A) = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{0}\}$

$m \times n$ matrix

Facts
1) $\text{col}(A)$ is a subspace of $\mathbb{R}^m$ (because a span is always a subspace)
2) $\text{row}(A)$ is a subspace of $\mathbb{R}^n$ (because a span is always a subspace)
3) $\text{Ker}(A)$ is a subspace of $\mathbb{R}^n$ (not obvious)

Null space

Facts
1) The resulting vectors will always be linearly dependent
2) $\dim(\ker A)$ = # of free variables
3) # of free vars + # of linked vars = total # of vars $= n$

Rank-Nullity Theorem

$$\dim(\text{null}(A)) + \text{rank}(A) = n \qquad (\text{rank}(A) = \dim(\text{col}(A)))$$

General Solution vs. Particular Solution

Goal: Solve $A\bar{x} = \bar{b}$

Idea: We can find all solutions $\bar{x}$ to $A\bar{x} = \bar{0}$, and then, find one solution to $A\bar{x}_0 = \bar{b}$

Theorem: $A(\bar{x} + \bar{x}_0) = \bar{b}$ gives the general solution to $A(\bar{x} + \bar{x}_0) = \bar{0}$

Ex. $A = \begin{pmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 10 \\ 3 \\ 4 \end{pmatrix}$ solve $A\bar{x} = \bar{b}$

Sol $\left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 10 \\ 0 & 1 & 1 & -2 & 3 \\ 1 & 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{\text{row op}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ so general sol is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4-s-t \\ 3-s+2t \\ s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\text{sol of } A\bar{x} = \bar{0}}$$

general homogeneous sol.

Thm Assume that $A\bar{x} = \bar{b}$ is a consistent system
1) If $\bar{x} = \bar{u}$ is a solution of $A\bar{x} = \bar{b}$ (inhomogeneous system) and $\bar{x} = \bar{v}$ is a solution of $A\bar{x} = \bar{0}$ (homogeneous system) then $\bar{u} + \bar{v}$ is a solution of $A\bar{x} = b$ (inhomogeneous system)
2) If $\bar{w}$ and $\bar{t}$ are solutions of $A\bar{x} = \bar{b}$ (inhomogeneous system) then $\bar{w} - \bar{t}$ is a solution of $A\bar{x} = \bar{0}$ (homogeneous system)

Thm Let A be a $m \times n$ matrix and $\bar{x} \in \mathbb{R}^n$. <u>The following are equivalent</u>

1) $A\bar{x} = \bar{b}$ is consistent for <u>every</u> $\bar{b} \in \mathbb{R}^m$
2) $\text{rank}(A) = m$
3) There are no zero rows in $\text{rref}(A)$
4) Every $\bar{b} \in \mathbb{R}^m$ is a lin. comb. of the columns of A
5) $\text{col}(A) = \mathbb{R}^m$
6) $\dim(\text{col}(A)) = m$

Thm Assume that $A\bar{x} = \bar{b}$ is consistent ($A, \bar{b}$ are fixed) <u>TFAE</u>

1) $A\bar{x} = \bar{b}$ has a unique solution
2) Every variable is linked (leading var.)
3) Every column has a leading 1 (pivot)
4) $A\bar{x} = \bar{0}$ has a unique solution
5) The columns of A are linearly independent
6) $\text{null}(A) = \ker(A) = \{\bar{0}\}$

$\left(\begin{array}{ccc|c} 1&0&0&1 \\ 0&1&0&1 \\ 0&0&1&1 \end{array}\right)$ unique

$\left(\begin{array}{ccc|c} 1&3&0&1 \\ 0&0&1&1 \\ 0&0&0&0 \end{array}\right)$ not unique (no pivot @ col 2)

$\left(\begin{array}{ccc|c} 1&0&0&1 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&0 \end{array}\right)$ unique

<u>Row Spaces</u>

The resulting (non-zero) vectors of $\text{rref}(A)$ form a basis of $\text{row}A$

↳ $\dim(\text{row}A) = \text{rank}(A)$

<u>Column Spaces</u>

To find a basis of $\text{col}A$, pick columns of A where $\text{rref}(A)$ has pivots

⚠ <u>WARNING</u> ☆ Need to pick columns of the original matrix (row op. affect $\text{col}A$)

↳ $\dim(\text{col}(A)) = \text{rank}(A)$

So. $\dim(\text{col}(A)) + \dim(\ker(A)) = n$ (rank-nullity theorem)

<u>Matrix Inverse</u>

The inverse of $A_{n\times n}$ is $B_{n\times n}$ where $AB = BA = I_n$

If such a matrix B exists, we call $B = A^{-1}$

Recipe for a 2x2 matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(A) = ad - bc$

- If $\det(A) = 0$ then A is not invertible
- If $\det(A) \neq 0$ then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Whenever there is a non-zero vector $\bar{x}$ with $A\bar{x} = \bar{0}$ then A is not-invertible
↳ Whenever the null sparce is non-trivial ($\neq \{\bar{0}\}$), A is not-invertible

<u>Fact</u> If A is invertible ($n \times n$). Then the system $A\bar{x} = \bar{b}$
1) is consistent
2) has a unique solution

<u>Properties of Inverse</u>
Assume A, C are invertible, then
1) A^{-1} is invertible, and $(A^{-1})^{-1} = A$
2) A^m is invertible, and $(A^m)^{-1} = (A^{-1})^m$ $(m \in \mathbb{N})$
3) A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$
4) kA is invertible, and $(kA)^{-1} = \frac{1}{k} A^{-1}$ $(k \neq 0)$
5) AC is invertible, and $(AC)^{-1} = C^{-1}A^{-1}$ (Note the order)

Inverse of an $n \times n$ matrix
A is invertible $\Leftrightarrow$ $\text{null}(A) = \{\bar{0}\}$ $\Leftrightarrow$ $\text{rank}(A) = n$
↳ $(A_{n \times n} | I_n) \underset{\text{row op}}{\rightsquigarrow} (I_n | A^{-1})$
If $\text{rank}(A) < n$ then A is not invertible

<u>Orthogonality</u>
A set $\{\bar{v}_1, \ldots, \bar{v}_m\}$ in $\mathbb{R}^n$ is called <u>orthogonal</u> if $\bar{v}_i \cdot \bar{v}_j = 0$ for any i, j
An orthogonal set is called <u>orthonormal</u> if all its vectors have length 1

Pythagorean Property: If $\{\bar{v}_1, \ldots, \bar{v}_m\}$ is orthogonal, then
$$\|\bar{v}_1 + \cdots + \bar{v}_m\|^2 = \|\bar{v}_1\|^2 + \cdots + \|\bar{v}_m\|^2 \qquad (\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}})$$

<u>Thm</u> If $\{\bar{v}_1, \ldots, \bar{v}_m\}$ is an orthogonal set of nonzero vectors, then it is L.I.

<u>Thm</u> Let $\{\bar{w}_1, \ldots, \bar{w}_m\}$ be an orthogonal <u>basis</u> of $\mathbb{R}^m$
Then for any $\bar{w} = \mathbb{R}^m$
$$\bar{w} = \left(\frac{\bar{w} \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1}\right) \bar{w}_1 + \left(\frac{\bar{w} \cdot \bar{w}_2}{\bar{w}_2 \cdot \bar{w}_2}\right) \bar{w}_2 + \cdots + \left(\frac{\bar{w} \cdot \bar{w}_m}{\bar{w}_m \cdot \bar{w}_m}\right) \bar{w}_m$$

"Fourier coefficients of $\bar{w}$"

Gram-Schmidt Process: Turn a basis into an orthogonal basis

Idea

Algorithm: Start from $\{\bar{v}_1, \dots, \bar{v}_n\}$ a basis of W

take $\bar{w}_1 = \bar{v}_1$

$\bar{w}_2 = \bar{v}_2 - proj_{v_1}\bar{v}_2$

$\bar{w}_3 = \bar{v}_3 - proj_{span(\bar{v}_1, \bar{v}_2)}\bar{v}_3 = \bar{v}_3 - proj_{\bar{w}_1}\bar{v}_3 - proj_{\bar{w}_2}\bar{v}_3$

$w_4 = \bar{v}_4 - proj_{span(v_1, v_2, v_3)}\bar{v}_4 = v_4 - proj_{\bar{w}_1}\bar{v}_4 - proj_{\bar{w}_2}\bar{v}_4 - proj_{\bar{w}_3}\bar{v}_4$

etc...

The result is an orthogonal basis of W

Complex Numbers: $\boxed{\sqrt{-1} = i}$

Euler's Formula: $\boxed{e^{i\pi} + 1 = 0}$

Properties:

length $z = a + bi$ so $|z|^2 = z \cdot \bar{z}$

$\bar{z} = a - bi$

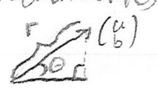
dividing $\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2}$ $(z \neq 0)$

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{cz} = c \cdot \bar{z}$
- $z = \bar{z} \iff z \in \mathbb{R}$
- $\bar{\bar{z}} = z$

- $|\bar{z}| = |z|$
- $|zw| = |z| \cdot |w|$
- $|z + w| \leq |z| + |w|$ (triangle inequality)

Geometry

Polar Coordinates: $\binom{a}{b} \sim \binom{r\cos\theta}{r\sin\theta}$



$r = \sqrt{a^2 + b^2}$

$\theta = \arctan\left(\frac{b}{a}\right)$

so $a + bi \sim r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta) = r \cdot e^{i\theta}$

If we have $z = re^{i\theta}$ and $w = se^{i\phi}$

$|z| = r$ $\quad |w| = s$

$zw = rse^{i\theta}e^{i\phi} = rse^{i(\theta + \phi)}$

$\Rightarrow |zw| = rs = |z| \cdot |w|$

The resulting number zw has norm rs and angle $\theta + \phi$

$\frac{1 + i}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = e^{i\frac{\pi}{4}}$

"Multiplying by $e^{i\pi/4}$ rotates by $\pi/4$"

$\nearrow \binom{\cos\theta}{\sin\theta}$

Note: The unit circle $\mathbb{C}$ is the set $\{|z| = 1\}$, so $\{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Determinants $A = n \times n$ matrix

$n=1$ $\det(a) = a$

$n=2$ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

A is invertible $\iff \det(A) \neq 0$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Ex $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{\text{use 4th row}}{=} -0\begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \overset{\text{using 3rd col}}{=} 3\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -3$

In general, can expand wrt any row or column

Properties:
- If A has a row (or columns) of 0's, then $\det A = 0$
- $\det(A^T) = \det A$
- The determinant of a triangle matrix is the product of diagonal entries

Fact If $A_{n \times n}$ is in RREF, then

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ a) $\text{rank}(A) = n$ and $\det A = 1$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ b) $\text{rank}(A) < n$ and $\det A = 0$

Fact (Row operations and determinants)
- Interchanging rows multiplies determinant by -1
- Multiplying a row by r multiplies determinant by $-1r$
- Adding a multiple of a row to another does not affect determinant

(can also work on columns by transposing)

Properties
- $\det(rA_{n \times n}) = r^n \det A$
- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(A) \leq 0 \iff A$ is ~~not~~ invertible
- If A is invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$

<u>Eigenvalues, Eigenvectors</u>

A = $n \times n$ matrix λ = scalar $\in \mathbb{R}$

Eigenvectors stay in the same direction when multiplied by A

$\boxed{A\bar{x} = \lambda \bar{x}}$

↳ λ is an eigenvalue, $\bar{x}$ is an eigenvector

In general, any non-zero element of $\ker A$ is an eigenvector with $\lambda = 0$

$\bar{x} \in \ker A \Leftrightarrow A\bar{x} = \bar{0} = 0 \cdot \bar{x}$

Finding eigenvalues and eigenvectors

$\bar{0} \neq \bar{x}$ is an eigenvector with eigenvalue $\lambda \Leftrightarrow A\bar{x} = \lambda\bar{x} \Leftrightarrow \lambda I_n \bar{x}$

$A\bar{x} - \lambda I_n \bar{x} = \bar{0} \Leftrightarrow \boxed{(A - \lambda I_n)\bar{x} = \bar{0}}$

<u>Fact</u> $\bar{x} = \bar{0}$ is an e.vect $\bar{w}$ e.val $\lambda \Leftrightarrow \boxed{x \in \ker(A - \lambda I_n)}$

$A\bar{x} = \lambda\bar{x} \Leftrightarrow (A - \lambda I_n)\bar{x} = \bar{0}$ (homogeneous system)

Need a <u>non-zero</u> solution to $\bar{x}$

↳ The only way is to have ∞ many solutions

i.e. $A - \lambda I_n$ should not be invertible

↳ $\underbrace{\det(A - \lambda I_n)}_{\text{characteristic polynomial}} = 0$

characteristic polynomial of degree n (has $\leq n$ roots)

The eigenvectors associated to λ, an e.val, are the solutions of $\underbrace{(A - \lambda I_n)}_{B}\bar{x} = \bar{0}$

B $(B\bar{x} = 0)$ $\rightarrow \ker(A - \lambda I_n)$ is the set of all such vectors

E_λ = eigenspace of λ

Let $A = n\times n$ matrix, λ an e.val

- The order of vanishing of the characteristic polynomial at λ is called the algebraic multiplicity

Ex $P(\lambda) = (\lambda-1)^5(x+2)^3$

5 = order of vanishing of 1 = algebraic multiplicity of $\lambda=1$

3 = order of vanishing of -2 = algebraic multiplicity of $\lambda=-2$

Note: If alg. mult. $=1$ then geom. mult. $=1$

- The geometric multiplicity of λ is the dimension of E_λ

Fact: geom. mult. of $\lambda \leq$ alg. mult. of λ (not always $=$)

Fact: eigenvectors from different eigenvalues are linearly independent

Goal: Find n L.I. eigenvectors $\rightsquigarrow$ they form a basis

- If this is possible, then we say A is diagonalizable

Fact: If A is diagonalizable, and $\bar{v}_1,\dots,\bar{v}_n$ forms a basis of $\mathbb{R}^n$ then $P = \begin{pmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{pmatrix}$ has the property $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{pmatrix}$ (where λ_i = e.val of $\bar{v}_i$)

What is this good for?

1) $\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A)$

$\quad\det(D) = \lambda_1\lambda_2\cdots\lambda_n$ so $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$

2) $P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^2 = PD^2P^{-1}$, $A^3 = PD^3P^{-1}$ $\Big\}\ A^n = PD^nP^{-1}$

Linear Transformations

Let U and V be vector spaces

A linear transformation T is a function from U to V such that

1) For any $\bar{u},\bar{v}\in U$, $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$
2) For any $c\in\mathbb{R}$, $T(c\bar{u}) = cT(\bar{u})$

Fact If $T: U\to V$ is linear then $T(\bar{0}) = \bar{0}$ (since $T(0\bar{u}) = 0T(\bar{u}) = \bar{0}$

Fact The only lin. trans from $\mathbb{R}^n \to \mathbb{R}^n$ are "matrix multi"

$T(x) = \underset{\text{standard matrix}}{A\bar{x}}$ where $A = \begin{pmatrix} | & | & & | \\ T(\bar{e}_1) & T(\bar{e}_2) & \cdots & T(\bar{e}_n) \\ | & | & & | \end{pmatrix}$

Rotation Matrix

$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$