MAT B41C W15 Find Solutions

254

1. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x - 2y + z = 0\}$. Which one of the following statements is true?

A. X is a subspace of $\mathbf{R}^3$ and $\dim X = 3$ ×

(B.) X is a subspace of $\mathbf{R}^3$ and $\{(2, 1, 0), (-1, 0, 1)\}$ is a basis of X

C. X is a subspace of $\mathbf{R}^3$ and $\{(2, 1, 0), (4, 2, 0)\}$ is a basis of X × — is dependent

D. X is a not a subspace of $\mathbf{R}^3$. ×

E. X is a line in $\mathbf{R}^3$ with direction vecor $(1, -2, 1)$ ×

F. X is a plane in $\mathbf{R}^3$ with normal $(2, 1, 0)$ ×

X is the plane through O with normal (1, -2, 1).

dim X = 2

∴ (B) is correct

2. Which two of the following statements are true?

I. $\{1, x, x^2\}$ is linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$. ✓ — is l.i.

II. A homogeneous linear system always has infinitely many solutions. NO — $[1 \mid 0]$ has a unique sol^n.

III. If A and B are 2×2 matrices, and A is invertible then $AB = 0$ implies $B = 0$. ✓

IV. If u and v are independent vectors in $\mathbf{R}^3$, then $\{u, v\} = \mathrm{span}\{u, v\}$. X ?

A. I and II.
(B.) I and III.
C. I and IV.
D. II and III.
E. II and IV.
F. III and IV.

$\{u,v\}$: Contains exactly 2 vectors

$\mathrm{span}\{u,v\}$: contains ∞ly many vectors

Contains $0 = 0u + 0v$, and so can't be l.i.

true!

$AB = 0$

$\Rightarrow A^{-1}(AB) = A^{-1}(0) = 0$

$\Leftrightarrow B = 0$.

3. Let A be a square $n \times n$ matrix with $n \geq 2$. Which of the following statements is true?

I. If rank $A = n - 1$, there is just one parameter in the general solution of $Ax = 0$.
II. If rank $A = n - 1$, there are $n - 1$ parameters in the general solution of $Ax = 0$.
III. If A is invertible, $Ax = 0$ has more than one solution
IV. If $Ax = 0$ has more than one solution, then rank $A < n$.

A. I. only
B. II. only
C. III only
D. I. and III.
E. I and IV.
F. III and IV.

4. If $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ and D is a $2 \times n$ matrix, then the second row of the matrix AD is

A. not defined unless $n = 2$.
B. twice the first row of D.
C. the same as the first row of D.
D. the same as the second row of D.
E. the sum of the first and the second rows of D.
F. the sum of twice the first row of D and the second row of D.

5. Let $S = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and consider the subset $W = \{A \in \mathbf{M}_{22} \mid SA = AS\}$. Which one of the following statements is true?

A. W is not a subspace of $\mathbf{M}_{22}$
B. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 4$
C. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 3$
(D) W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 2$
E. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 1$
F. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 0$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 2a \\ c & 2c \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\therefore W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{array}{l} a = a \\ c = 0 \\ 2a = b \\ 2c = 0 \end{array} \right\}$ $\therefore W$ is a ss of M_{22}.

$= \left\{ \begin{bmatrix} a & 2a \\ 0 & d \end{bmatrix} \middle| \ a, d \in \mathbb{R} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\therefore \dim W = 2$

6. Find the value of t for which $(1, 3, t)$ belongs to $\operatorname{span}\{(1, 2, 1), (1, 1, 2)\}$.

A. -2
B. -1
(C) 0
D. 1
E. 2
F. 7

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & t \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & t-1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & t \end{array}\right]$

This is consistent $\Leftrightarrow t = 0$

7. For a non-homogeneous system of 15 equations in 12 unknowns, answer the following three questions:

- Can the system be inconsistent? ✓
- Can the system have a unique solution? ✓
- Can the system have infinitely many solutions? ✓

12 × 15 [| b] , b ≠ 0

A. No, Yes, Yes.
B. Yes, Yes, No.
C. Yes, No, Yes.
D. No, No, Yes.
(E.) Yes, Yes, Yes.
F. No, Yes, No.

eg 15 [0 | 1 0 ⋮ 0] (12) is inconsistent

[1 0 ⋯ 0 / 0 | 1 0 ⋮ 0] has ∞ly many soln with 11 parameters

eg [I_{12} | 1 0 ⋮ ; 0 | 0] has a unique soln
(0: size 3×12)

8. Let A be a square $n \times n$ matrix. Which one of the statements below **is not equivalent** to

"The rows of A are linearly independent" (so A is invertible)

X (A.) 0 is an eigenvalue of A (so $Ax = 0x$ has a non-zero soln!)

B. The homogeneous system $Ax = 0$ has a unique solution ✓

C. The columns of A form a basis of $\mathbf{R}^n$ ✓

D. rank $A = n$ ✓

E. det $A \neq 0$ ✓

F. A is invertible ✓

(A |

9. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3$, find

$$\begin{vmatrix} 3a & 3d & 3g \\ b+5c & e+5f & h+5i \\ -2c & -2f & -2i \end{vmatrix} = 3(-2) \begin{vmatrix} a & d & g \\ b+5c & e+5f & h+5i \\ c & f & i \end{vmatrix}$$

A. -64
B. -18
C. -16
D. 16
(E) 18
F. 64

$$= -6 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \quad (\text{having used } -5R_3 + R_2 \to R_2)$$

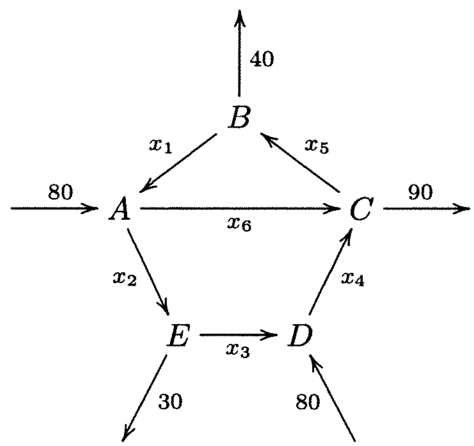
$$= -6 \det A^t = -6 \cdot (-3) = 18$$

10. The vectors $u_1 = (1, -1, 2)$, $u_2 = (-5, -1, 2)$, and $u_3 = (0, 2, 1)$ form an orthogonal basis of $\mathbf{R}^3$. If we write $v = (1, 0, -1) = a_1u_1 + a_2u_2 + a_3u_3$, what is a_2?

A. $-2/6$
B. $2/6$
(C) $-7/30$
D. $7/30$
E. $-1/5$
F. $1/5$

$$a_2 = \frac{v \cdot u_2}{\|u_2\|^2} = \frac{-7}{25+1+4} = \frac{-7}{30}$$

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables x_i, $i = 1, \ldots, 6$.
(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection	Flow in	=	Flow out
A	$80 + x_1$	=	$x_2 + x_6$
B	x_5	=	$x_1 + 40$
C	$x_4 + x_6$	=	$90 + x_5$
D	$x_3 + 80$	=	x_4
E	x_2	=	$x_3 + 30$

2

Constraints: $x_i \geq 0$ (one-way streets)
$x_i \in \mathbb{Z}$ (# of cars)

1

11(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$x_1 = -40 + s$
$x_2 = 40 + s - t$
$x_3 = 10 + s - t$
$x_4 = 90 + s - t$; $s, t \in \mathbb{R}$
$x_5 = s$
$x_6 = t$

2

c) If $\overline{AC}$ were closed due to roadwork, find the <u>minimum</u> flow along $\overline{ED}$, **using your results from (b).**
(You must justify all your answers.)

$\overline{AC}$ closed $\Leftrightarrow x_6 = t = 0$. Flow along $\overline{ED}$ is $x_3 = 10 + s$

$x_1 \geq 0 \Leftrightarrow s \geq 40$
$x_2 \geq 0 \Leftrightarrow -40 \leq s$
$x_3 \geq 0 \Leftrightarrow -10 \leq s$
$x_4 \geq 0 \Leftrightarrow -90 \leq s$
$x_5 \geq 0 \Leftrightarrow 0 \leq s$

$\Rightarrow s \geq 40$

Hence the minimum flow along $\overline{ED}$ is 50

$\frac{1}{2} + \frac{1}{2}$

12. Let $X = \text{span}\{(1,-1,1,0),(0,1,1,1),(1,2,4,3),(1,0,2,2)\} = \{(x,y,z,w) \mid -2x-y+z=0\}$

$z = 2x+y$

2 a) Find any basis for X, and hence find $\dim X$.

1.5 b) Find a basis for X which is a **subset** of the given spanning set above. .5 "subset" + 1

1.5 c) Extend your basis for X in part (b) to a basis of $\mathbf{R}^4$. .5 method + 1 correct + just

1 d) If X were the rowspace of a 4×4 matrix A, how many parameters would there be in the general solution to $Ax = 0$? .5 + .5

(a) & (b)

Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} (1) & 0 & 1 & 1 \\ 0 & (1) & 3 & 1 \\ 0 & 0 & 0 & (1) \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(so $X = \text{col } A$)

Hence $\{\underset{v_1}{(1,-1,1,0)}, \underset{v_2}{(0,1,1,1)}, \underset{v_4}{(1,0,2,2)}\}$ is a basis for X which answers both (a) & (b). Hence $\dim X = 3$ (1)

(c) We pick $v \in \mathbb{R}^4$ s.t. rank $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ & v & & \end{bmatrix} = 4$

But $\begin{bmatrix} v_1 \\ v_2 \\ v_4 \\ v \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ & v & & \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ & v & & \end{bmatrix} \sim \begin{bmatrix} (1) & -1 & 1 & 0 \\ 0 & (1) & 1 & 0 \\ 0 & 0 & 0 & (1) \\ & v & & \end{bmatrix}$

Hence if we add $v = (0,0,1,0)$ to the vectors above, we obtain a basis of $\mathbb{R}^4$.

(d) If $X = \text{row}(A)$, then rank $A = \dim \text{row}(A) = 3$, so there would be $4 - 3 = 1$ parameter in the general soln to $Ax = 0$

829

13. Let $W = \{(x, y, z, w) \in \mathbf{R}^4 \mid x - z - w = 0\}$. $x = z + w$

.5 a) *Without referring to the Subspace Test* briefly explain why W is a subspace of $\mathbf{R}^4$.

1.5 b) Find a basis for W. (.5 – justn)

2 c) Use the **Gram-Schmidt algorithm** to find an orthogonal basis for W.

1 correct alg
.5 vectors are orthog
.5 vector belong to W

2 d) Find the best approximation by a vector in W to the vector $(1, 0, 1, 1)$.

.5 proj'n +.5 correct proj form +.5 ∈ W +.5 correct

a) $W = \ker [1\ 0\ -1\ -1]$ and hence is a subspace of $\mathbb{R}^4$

b) From A, $W = \{(s+t, r, s, t) \mid r, s, t \in \mathbb{R}\}$ ∴

$\{(1,0,1,0), (1,0,0,1), (0,1,0,0)\}$ is a basis for W. (v_1, v_2, v_3)

c) Set $w_1 = v_1 = (1,0,1,0)$

$w_2' = v_2 - \left(\frac{v_2 \cdot w_1}{\|w_1\|^2}\right) w_1 = (1,0,0,1) - \frac{1}{2}(1,0,1,0)$

$= (\frac{1}{2}, 0, -\frac{1}{2}, 1)$

Let's use $w_2 = (1, 0, -1, 2)$.

1 0 0 1
0 1 0 0
1 0 2 -1

Then $w_3 = v_3 - \left(\frac{v_3 \cdot w_1}{\|w_1\|^2}\right) w_1 - \left(\frac{v_3 \cdot w_2}{\|w_2\|^2}\right) w_2$

$= (0,1,0,0) - 0w_1 - 0w_2$

$= (0,1,0,0)$

∴ $\{(1,0,1,0), (1,0,-1,2), (0,1,0,0)\}$ is an orthogonal basis of W.

d) This is $\text{proj}_W(1,0,1,1) = \frac{(1,0,1,1)\cdot(1,0,1,0)}{2}(1,0,1,0) + \frac{(1,0,1,1)\cdot(1,0,-1,2)}{6}(1,0,-1,2)$

$+ \frac{(1,0,1,1)\cdot(0,1,0,0)}{1}(0,1,0,0) = (1,0,1,0) + \frac{1}{3}(1,0,-1,2)$

$= (\frac{4}{3}, 0, \frac{2}{3}, \frac{2}{3})$.

14. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

1 a) Find the characteristic polynomial of A, and use this to show that the eigenvalues of A are 2 and -1.

2 b) Find a basis of $E_2 = \{v \in \mathbf{R}^3 \mid Av = 2v\}$.

1 c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

2 d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If this is not possible, explain why.

a) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} \overset{-R_1+R_3 \to R_3}{=} \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -2+\lambda & 0 & 2-\lambda \end{vmatrix} \overset{C_1+C_3\to C_3}{=} \begin{vmatrix} 1-\lambda & -1 & -\lambda \\ -1 & 1-\lambda & -2 \\ -2+\lambda & 0 & 0 \end{vmatrix}$

$\overset{\text{row } 3}{=} (\lambda-2)\begin{vmatrix} -1 & -\lambda \\ 1-\lambda & -2 \end{vmatrix} = (\lambda-2)\{2-\lambda^2+\lambda\} = -(\lambda-2)^2(\lambda+1)$.

$(2-\lambda)(1+\lambda)$ $\quad (-4+3\lambda^2-\lambda^3)$

$-(\lambda^3-3\lambda^2+4)$

$-\lambda^3+3\lambda^2-4$

Hence evals are 2 and -1.

b) $E_2 = \ker(A-2I) = \ker\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-s-t, s, t) \mid s,t \in \mathbb{R}\}$ 1

$\therefore \{(-1,1,0), (-1,0,1)\}$ is a basis for E_2 (.5, .5)

c) $E_{-1} = \ker(A+I) = \ker\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ .5

$= \{(s,s,s)\}$ $\therefore$ $\{(1,1,1)\}$ is a basis for E_{-1} .5

d) Let $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (1) and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ($\frac{1}{2}$). Then $P^{-1}AP = D$

and P is invertible because $\dim E_2 + \dim E_{-1} = 3$. $\frac{1}{2}$

347

15. State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

a) If A is an 4×3 matrix and if a row echelon form of A has a row of zeros, then rank $A < 3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank } 3.$$

ANSWER FALSE

b) The dimension of the kernel of the matrix $[1 \quad 2 \quad 3 \quad 4 \quad 5]$ is 4.

$$\dim \ker [1\ 2\ 3\ 4\ 5] = \#\text{cols} - \text{rank}[1\ 2\ 3\ 4\ 5]$$
$$= 5 - 1$$
$$= 4$$

ANSWER TRUE

15 (cont.)

c) If $\{v_1, v_2\}$ is linearly independent in a vector space V, then $\{v_1 - v_2, v_1 + 2v_2\}$ is also linearly independent.

Suppose $a(v_1 - v_2) + b(v_1 + 2v_2) = 0$. Then

$(a+b)v_1 + (2b-a)v_2 = 0$. Since $\{v_1, v_2\}$ is l.i.,

$$\left.\begin{aligned} a + b &= 0 \\ -a + 2b &= 0 \end{aligned}\right\} \Rightarrow 3b = 0 \Rightarrow b = 0, \text{ so } a = 0 \text{ for}$$

$\therefore \{v_1 - v_2, v_1 + 2v_2\}$ is l.i.

ANSWER TRUE

d) The funtion $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x^2, x + y)$ is a linear transformation.

Note $T(1,0) = (1,1)$

but $T(2,0) = (4,2) \neq 2 \cdot T(1,0)$

ANSWER FALSE

16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.

In what follows, A denotes an $n \times n$ matrix.

a) Prove that if u, v and w are eigenvectors of A corresponding to three distinct eigenvalues, then $\{u, v, w\}$ is linearly independent.

Suppose $Au = \lambda_1 u$, $Av = \lambda_2 v$, $Aw = \lambda_3 w$ $\lambda_1 \neq \lambda_2$, $\lambda_2 \neq \lambda_3$, $\lambda_1 \neq \lambda_2$

Suppose $au + bv + cw = 0$ (1). Applying A we obtain

$a\lambda_1 u + b\lambda_2 v + c\lambda_3 w = 0$ (2)

Then (2) - λ_1(1) yields $b(\lambda_2 - \lambda_1)v + c(\lambda_3 - \lambda_1)w = 0$ (3)

Applying A to (3), we obtain $b(\lambda_2 - \lambda_1)\lambda_2 v + c(\lambda_3 - \lambda_1)\lambda_3 w = 0$ (4)

Then (4) - λ_2(3) yields $c(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)w = 0$. Hence $c = 0$ (as $w \neq 0$).

As $v \neq 0$, (3) implies $b = 0$. Then (1) implies $a = 0$ (as $u \neq 0$).

$\therefore \{u, v, w\}$ is l.i.

b) Prove that if all the eigenvalues of A are non-zero, then A is invertible.

A is invertible $\Leftrightarrow$ $\det A \neq 0$. If $\lambda = 0$ is not an eigenvalue, then $\det(A - 0I) \neq 0$, i.e. $\det A \neq 0$.

Hence A is invertible

OR If all evals are not zero, then

$AX = 0X \Rightarrow X = 0$, So A is invertible.