University of Ottawa
Department of Mathematics and Statistics

MAT1322

Calculus II

Midterm test 3

March 25, 2019

Instructor: Vadim Kaimanovich

Duration: 75 minutes

Read the following information before starting the test:

- Verify that your copy of the test contains 5 pages, including this one.
- Write your name and student number on this page.
- Work the problems in the space provided. Use the back-pages and the blank sheet attached at the end for rough work. Do not use any other paper. Before submitting the test remove the rough work page 5.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.
- Circle your final answers.

The Faculty of Science requires that you read and sign the following statement:

Cellular phones, calculators or other electronic devices and course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the test.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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Problem	1(2)	2(2)	3(2)	4(2)	5(2)	6(2)	Total(12)
Points							

1. Find the convergence radius R and the convergence interval I of the power series $\sum_{n=0}^{\infty} \frac{(2x+3)^n}{4^n\sqrt{n+1}}$.

A. $R=0, I=[-7/2, 1/2]$ B. $R=1, I=(-7/2,1/2]$ C. $R=2, I=(1/2,7/2]$ D. $R=1, I=(1/2,7/2)$ E. $R=-2, I=(-7/2,1/2]$ F. $R=2, I=[-7/2,1/2)$ G. none of the above

Put $a_n = \frac{(2x+3)^n}{4^n\sqrt{n+1}}$, then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x+3|}{4}\frac{\sqrt{n+1}}{\sqrt{n+2}} \to \left|\frac{2x+3}{4}\right|$, so that the endpoints of the convergence interval are the solutions of $\left|\frac{2x+3}{4}\right|=1$, i.e., $2x+3=\pm 4$, whence $x_1=-\frac{7}{2}$, $x_2=\frac{1}{2}$, the length $2R$ of the convergence interval is 4, so that $R=2$

For $x=x_1$ $a_n=\frac{(-1)^n}{\sqrt{n+1}}$, and the series converges by the alternating test

For $x=x_2$, $a_n=\frac{1}{\sqrt{n+1}}$, and the series diverges (for instance, by the integral test)

2. By using differentiation or integration of power series find the power series representation of the function $f(x) = 3/(1+2x)^2$ at the point 0. What are its convergence interval I and the second term a_2?

A. $I=(-1,2), a_2=9x^2$ B. $I=(1,2), a_2=36x^2$ C. $I=(1,2), a_2=-9x^2$ D. $I=(-1,2), a_2=36x^2$ E. $I=(-1/2,1/2), a_2=36x^2$ F. $I=(-1/2,1/2), a_2=9x^2$ G. none of the above

$\int f(x)\,dx = \int \frac{3}{(1+2x)^2}\,dx = -\frac{1}{2}\frac{3}{1+2x}+C$, so that $f(x)=g'(x)$ for $g(x)=-\frac{3}{2}\frac{1}{1+2x}$ whose power representation is $g(x)=-\frac{3}{2}\frac{1}{1+2x} = -\frac{3}{2}\left(1-2x+(2x)^2-(2x)^3+\dots\right)$
$= -\frac{3}{2}+3x-6x^2+12x^3-24x^4+\dots = \sum_{n=0}^{\infty} 3(-2)^{n-1}x^n$,
whence inside the convergence interval of g (i.e., for $|2x|<1$)
$f(x)=g'(x)=3-12x+36x^2-96x^3+\dots = \sum_{n=0}^{\infty} 3(n+1)\cdot(-2x)^n$,
which diverges at the endpoints of the convergence interval (i.e., for $x=\pm\frac{1}{2}$).

3. What is the third order entry a_3 of the Taylor series of the function $f(x) = \sqrt{1+2x}$ at the point 0? By using the Taylor series find the limit $L = \lim_{x\to 0}(\sqrt{1+2x} - 1 - x)/x^2$.

A. $a_3 = x^2/2, L = -1/2$ B. $a_3 = x^3/2, L = -1/2$ C. $a_3 = x^3/2, L = 1/2$ D. $a_3 = -x^3/2, L = 1/2$ E. $a_3 = x^2/2, L = 1/2$ F. $a_3 = x^3, L = 1$ G. none of the above

The derivatives of the function $f(x) = (1+2x)^{1/2}$ are

$f'(x) = 2\cdot\frac{1}{2}\cdot(1+2x)^{-1/2} = (1+2x)^{-1/2}$,

$f''(x) = 2\cdot\left(-\frac{1}{2}\right)\cdot(1+2x)^{-3/2} = -(1+2x)^{-3/2}$,

$f'''(x) = -2\cdot\left(-\frac{3}{2}\right)(1+2x)^{-5/2} = 3(1+2x)^{-5/2}$, so that

$f(0) = 1$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 3$, whence

the Taylor series is $1 + x - \frac{x^2}{2} + \frac{3x^3}{6} + \ldots$

$= 1 + x - \frac{x^2}{2} + \frac{x^3}{2} + \ldots$, so that $a_3 = \frac{x^3}{2}$, and

$\frac{f(x) - 1 - x}{x^2} = -\frac{1}{2} + \frac{x}{2} + \ldots$, so that $L = -\frac{1}{2}$

4. For which of the following differential equations and initial conditions is the function $y(t) = (1 - 2e^{2t})/(1 + 2e^{2t})$ a solution?

A. $y' = y^2, y(0) = 1/3$ B. $y' = y^2+1, y(0) = 1/3$ C. $y' = y^2-1, y(0) = -1/3$ D. $y' = y^2, y(0) = -1/3$ E. $y' = y^2+1, y(0) = -1/3$ F. $y' = y^2-1, y(0) = 0$ G. none of the above

$$y'(t) = \frac{(1-2e^{2t})'(1+2e^{2t}) - (1-2e^{2t})(1+2e^{2t})'}{(1+2e^{2t})^2}$$

$$= \frac{-4e^{2t}(1+2e^{2t}) - 4e^{2t}(1-2e^{2t})}{(1+2e^{2t})^2}$$

$$= \frac{-8e^{2t}}{(1+2e^{2t})^2}, \text{ whence}$$

$y^2 - y' = \frac{(1-2e^{2t})^2 + 8e^{2t}}{(1+2e^{2t})^2} = 1$, so that $y' = y^2 - 1$;

$$y(0) = \frac{1-2}{1+2} = -\frac{1}{3}$$

5. Solve the differential equation

$$\frac{dy}{dx} = 1 + y + x^2 + x^2 y$$

with the initial condition $y(0) = -2$.

A. $y(x) = e^{x+x^3/3} - 1$ B. $y(x) = -e^{x+x^3/3} - 1$ C. $y(x) = -e^{x+x^3/3} + 1$ D. $y(x) = -x - x^3/3 + 2$
E. $y(x) = x - x^3/3 - 2$ F. $y(x) = -e^{-x+x^3/3} - 1$ G. $y(x) = x + x^3/3 - 2$ H. none of the above

$\frac{dy}{dx} = (1+x^2)(1+y)$, i.e. $\frac{dy}{1+y} = (1+x^2)\,dx$, so that $\int \frac{dy}{1+y} = \int (1+x^2)\,dx + C$, whence $\ln|1+y| = x + \frac{x^3}{3} + C$, and $|1+y| = e^c e^{x+\frac{x^3}{3}}$ so that the general solution is $1+y = K e^{x+\frac{x^3}{3}}$ (with $K \in \mathbb{R}$), i.e., $y = Ke^{x+\frac{x^3}{3}} - 1$, and $y(0) = K-1$, whence $K-1=-2$, and $K=-1$

6. Which of the following functions $f = f(x,y)$ have the property that $f_{xx} + f_{yy} = 0$: (I) $\sin(xy)$, (II) $x^3 - 3xy^2$, (III) $\ln(x^2 - y^2)$, (IV) $e^{-x}\cos y - e^y \cos x$?

A. (II) and (IV) B. (III) and (IV) C. (I), (II) and (IV) D. (II), (III) and (IV) E. (I) and (III) F. none
G. all H. none of the above

(I): $f_x(x,y) = y\cos(xy)$, $f_{xx}(x,y) = -y^2 \sin(xy)$; in the same way, $f_{yy}(x,y) = -x^2\sin(xy)$, so that $f_{xx} + f_{yy} = (-x^2 - y^2)\sin(xy) \neq 0$

(II) $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = -6x \Rightarrow f_{xx} + f_{yy} = 0$

(III) $f_x(x,y) = \frac{2x}{x^2-y^2}$, $f_{xx}(x,y) = \frac{2(x^2-y^2) - 4x^2}{(x^2-y^2)^2} = \frac{-2x^2-2y^2}{(x^2-y^2)^2}$

$f_y(x,y) = \frac{-2y}{x^2-y^2}$, $f_{yy}(x,y) = \frac{-2(x^2-y^2) - 4y^2}{(x^2-y^2)^2} = \frac{-2x^2-2y^2}{(x^2-y^2)^2}$

$\Rightarrow f_{xx} + f_{yy} = -4(x^2+y^2)/(x^2-y^2)^2 \neq 0$

(IV) $f_{xx}(x,y) = e^{-x}\cos y + e^y \cos x$, $f_{yy}(x,y) = -e^{-x}\cos y - e^y\cos x$ $\Big\}$ $f_{xx} + f_{yy} = 0$

extra page for calculations (please remove it when submitting the test!)