Version A

Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

DISCRETE MATH FOR COMPUTING Instructor: ELIZABETH MALTAIS

MAT1348A – Midterm 2 – Monday, March 18, 2019

- ▷ Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:
- ▷ This is a 75-minute **closed-book** test. No notes. No calculators.
- ▷ The exam consists of 7 pages.
  Page 7 contains the **Table of Important Set Identities**.
- ▷ The maximum points possible = **31 points**.
- ▷ Questions 1–3 are **short-answer** problems worth a total of **18 points**. Write your answer in the appropriate box, space, or circle your answer. You do not need to give detailed explanations.
- ▷ Questions 4–6 are **long-answer** problems worth a total of **13 points**. To receive full marks, your solution must be complete, correct, and contain all relevant details.
- ▷ Read all questions carefully and be sure to follow the instructions for the individual problems. You may ask for reasonable clarifications.
- ▷ For additional work space, you may use the backs of pages.
  Do not use any of your own scrap paper.
- ▷ You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read and understood, and will comply with the above instructions.**

FAMILY NAME:	STUDENT NUMBER:
FIRST NAME:	†SIGNATURE:

SOLUTIONS

Do not write in this table.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Total
Maximum points	8 pts	6 pts	4 pts	4 pts	5 pts	4 pts	31 points
Marks obtained							

Short-answer Questions.

Write your final answer in the answer box, the space provided, or circle your answer.

Q1. [**8 points**] Let $S = \Big\{a, \{a\}, \{a,b\}, \{a,\{b\}\}\Big\}$ and $T = \Big\{a, \{a\}, \{b\}\Big\}$.

a. Compute the following cardinalities.

$|S \times T| = |S| \cdot |T| = 4 \cdot 3 = 12$

$|\mathcal{P}(T)| = 2^{|T|} = 2^3 = 8$

$|S \cup T| = \big|\{a, \{a\}, \{b\}, \{a,b\}, \{a,\{b\}\}\}\big| = 5$

b. Give each of the following sets in list notation (use set braces $\{ \;\; \}$ where appropriate).

$S \cap T = \{a, \{a\}\}$

$T - S = \{\{b\}\}$

$S \oplus T = \{\{a,b\}, \{a,\{b\}\}, \{b\}\}$

c. True or false? Circle your answer. You do not need to justify.

$\{\{a\}, \{b\}\} \subseteq \mathcal{P}(T)$ TRUE FALSE

$\{b\} \not\subseteq T \Rightarrow \{b\} \notin \mathcal{P}(T) \Rightarrow \{\{a\},\{b\}\} \not\subseteq \mathcal{P}(T)$

$\{a,b\} \in \mathcal{P}(S)$ TRUE FALSE

$b \notin S \Rightarrow \{a,b\} \not\subseteq S \Rightarrow \{a,b\} \notin \mathcal{P}(S)$

$(\underbrace{a}_{\in T}, \underbrace{\{a\}}_{\in T}) \in T \times T$ TRUE FALSE

$(\emptyset, \emptyset) \in \mathcal{P}(S) \times \mathcal{P}(S)$ TRUE FALSE

$\emptyset \subseteq S \Rightarrow \emptyset \in \mathcal{P}(S) \Rightarrow (\emptyset, \emptyset) \in \mathcal{P}(S) \times \mathcal{P}(S)$

Q2. [**6 points**] Consider the following two functions:

$$h : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z} \qquad g : \mathbb{R} \to \mathbb{R}$$
$$h(r, s) = r + s \qquad g(t) = 8t^2$$

Circle the correct response. You do not need to justify your answer.

Is h injective (one-to-one)? **YES** **NO**

Is h surjective (onto)? **YES** **NO**

Is g injective (one-to-one)? **YES** **NO**

Is g surjective (onto)? **YES** **NO**

Is the composition $g \circ h$ defined? **YES** **NO**

Is the composition $h \circ g$ defined? **YES** **NO**

Q3. [**4 points**] Let $A = \{1, 2, 3, 4, 5, 6, 10, 12\}$ and let $\mathcal{R}$ be the relation on A defined by the following rule:

for all $u, v \in A$, $\quad u \mathcal{R} v \quad$ if and only if $\quad u$ divides v.

a. List every element $a \in A$ for which 3 is related to a by $\mathcal{R}$, that is, for which $3 \mathcal{R} a$.

$3 \mathcal{R} 3, 3 \mathcal{R} 6, 3 \mathcal{R} 12$

b. Give one element $x \in A$ for which 3 is not related to x by $\mathcal{R}$, that is, for which $3 \not\mathcal{R} x$.

$3 \not\mathcal{R} 2$

c. Is $\mathcal{R}$ a reflexive relation on A? *Circle:* **YES** **NO**

If you circled **NO**, give a counterexample below:

d. Is $\mathcal{R}$ a symmetric relation on A? *Circle:* **YES** **NO**

If you circled **NO**, give a counterexample below:

$5 \mathcal{R} 10$ but $10 \not\mathcal{R} 5$

Long-Answer Questions.

Detailed solutions with clear explanations are required.

Q4. [**4 points**] Let X and Y denote subsets of a universal set $\mathcal{U}$. The goal of this question is to determine whether or not the following equation is true for all sets X and Y.

$$\left(\overline{X} \cup Y\right) - \left(X - \overline{Y}\right) = \overline{X}$$

If you circle **YES**, then you must prove it using a membership table, or another method of your choice. If you circle **NO**, then you must provide an explicit counterexample (that is, give actual sets X and Y that certify that this equation can be false), and justify why it is a counterexample.

Is $(\overline{X} \cup Y) - (X - \overline{Y}) = \overline{X}$ a true set identity? *Circle*: YES NO

Justification:

Using membership table:

X	Y	$\overline{X} \cup Y$	$X - \overline{Y}$	$(\overline{X} \cup Y) - (X - \overline{Y})$	$\overline{X}$
1	1	1	1	0	0
1	0	0	0	0	0
0	1	1	0	1	1
0	0	1	0	1	1

since membership is the same for these two sets, they are equal.

Using Table of Important Set Identities:

Left set $= (\overline{X} \cup Y) - (X - \overline{Y})$

$= (\overline{X} \cup Y) - (X \cap \overline{\overline{Y}})$ (difference law)

$= (\overline{X} \cup Y) - (X \cap Y)$ ((double) complementation law)

$= (\overline{X} \cup Y) \cap (\overline{X \cap Y})$ (difference law)

$= (\overline{X} \cup Y) \cap (\overline{X} \cup \overline{Y})$ (De Morgan's law)

$= \overline{X} \cup (Y \cap \overline{Y})$ (Distributive Law)

$= \overline{X} \cup \emptyset$ (Complement law)

$= \overline{X}$ (Identity Law)

$=$ Right Set

Q5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined as follows: **[5 points]**

$$f(s,t) = (1+3t,\, 2s-t)$$

Prove that f is a **bijection**.

For each step in your proof, make sure it is clear whether it is an assumption, something you are about to prove, or something that follows from a previous step or definition. If you use any variables in your proof, make sure you clearly state what they represent.

We must prove that f is injective and surjective.

[injective]. Let $(a,b),(c,d) \in \mathbb{R}^2$. Assume $f(a,b)=f(c,d)$ (goal: prove $(a,b)=(c,d)$).

Then $(1+3b, 2a-b) = (1+3d, 2c-d)$ (by f's rule)

$\Rightarrow 1+3b = 1+3d$ and $2a-b = 2c-d$

$\Rightarrow b=d$ hence $2a-b = 2a-d = 2c-d$

$\Rightarrow a=c$ So $(a,b)=(c,d)$ (goal!)

we proved $(f(a,b)=f(c,d)) \to ((a,b)=(c,d))$ $\therefore$ f is injective.

[Surjective]. Let $(y,z) \in \mathbb{R}^2$. (goal: find $(a,b)\in\mathbb{R}^2$ such that $f(a,b)=(y,z)$).

$\Rightarrow$ we want $f(a,b)=(y,z)$ $\Rightarrow$ want $(1+3b, 2a-b) = (y,z)$

$\Rightarrow$ want $1+3b=y$ and $2a-b=z$

$\Rightarrow$ want $b=\frac{y-1}{3}$ and so we want $2a = z+b = z+\frac{y-1}{3}$

$\Rightarrow$ want $a = \frac{z}{2}+\frac{y-1}{6}$

(or $a=\frac{3z+y-1}{6}$)

To achieve our goal, we need $(a,b)=\left(\frac{3z+y-1}{6}, \frac{y-1}{3}\right)$.

Since $a,b \in \mathbb{R}$, it follows that $\frac{3z+y-1}{6} \in \mathbb{R}$, and $\frac{y-1}{3} \in \mathbb{R}$ $\therefore (a,b)\in\mathbb{R}^2$.

Moreover, $f(a,b) = f\left(\frac{3z+y-1}{6}, \frac{y-1}{3}\right)$

$= \left(1+3\left(\frac{y-1}{3}\right), 2\left(\frac{3z+y-1}{6}\right)-\left(\frac{y-1}{3}\right)\right)$

$= \left(\frac{3+3y-3}{3}, \frac{6z+2y-2-2(y-1)}{6}\right)$

$= (y,z)$ $\therefore$ f is surjective

Q6. [4 points] Let $\mathcal{R}$ be a relation on the set $\mathbb{Z}^+$ defined by the following rule:

$$\text{for all } x, y \in \mathbb{Z}^+, \qquad x \mathcal{R} y \qquad \text{if and only if} \qquad \frac{x}{y} = 2^k \quad \text{for some integer } k.$$

For example, $3 \mathcal{R} 3$ because $\frac{3}{3} = 1 = 2^0$ and $0 \in \mathbb{Z}$, and $3 \mathcal{R} 6$ because $\frac{3}{6} = \frac{1}{2} = 2^{-1}$ and $-1 \in \mathbb{Z}$.

(a) Prove that $\mathcal{R}$ is a **transitive** relation on $\mathbb{Z}^+$.

For each step in your proof, make sure it is clear whether it is an assumption, something you are about to prove, or something that follows from a previous step or definition. If you use any variables in your proof, make sure you clearly state what they represent.

proof. Let $x, y, z \in \mathbb{Z}^+$.

Assume $x \mathcal{R} y$ and $y \mathcal{R} z$ (goal: prove $x \mathcal{R} z$)

Then $\frac{x}{y} = 2^k$ for some integer $k \in \mathbb{Z}$ and $\frac{y}{z} = 2^l$ for some integer $l \in \mathbb{Z}$

Thus $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = 2^k \cdot 2^l = 2^{k+l}$.

Since $k, l \in \mathbb{Z}$, it follows that $k + l \in \mathbb{Z}$. $\therefore$ $x \mathcal{R} z$ (by $\mathcal{R}$'s rule).

We proved $(x \mathcal{R} y \wedge y \mathcal{R} z) \rightarrow (x \mathcal{R} z)$ $\therefore$ $\mathcal{R}$ is transitive.

(b) Now, let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 12, 16, 18, 24, 28, 36\}$. Let $\mathcal{S}$ be a relation on the set A defined by the following rule:

$$\text{for all } x, y \in A, \qquad x \mathcal{S} y \qquad \text{if and only if} \qquad \frac{x}{y} = 2^k \quad \text{for some integer } k.$$

Fact: The relation $\mathcal{S}$ is an **equivalence relation** A. You do **not** need to prove this! Simply give the **equivalence class** of 24 with respect to $\mathcal{S}$. For this part, you do not need to justify your answer.

Answer:

$[24]_{\mathcal{S}} = \{3, 6, 12, 24\}$

$\frac{24}{3} = 2^3$, $\frac{24}{6} = 2^2$, $\frac{24}{12} = 2^1$ $\frac{24}{24} = 2^0$.

1. 2.	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
3. 4.	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination Laws
5. 6.	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8. 9.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
10. 11.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
12. 13.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
14. 15.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
16. 17.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
18. 19.	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	Complement Laws
20.	$A - B = A \cap \overline{B}$	Difference Law
21. 22.	$A \oplus B = (A - B) \cup (B - A)$ $A \oplus B = (A \cup B) - (A \cap B)$	Symmetric Difference Laws

(end of Midterm 2) — you may detach this page