Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Math for Computing — Instructor: Elizabeth Maltais

MAT1348A – Midterm 1 – Monday, February 11, 2019

- Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:
- This is a 75-minute **closed-book** test. No notes. No calculators.
- **Put away everything** except for a few pens or pencils, an eraser, and your student id. card.
- The exam consists of 9 pages.**Page 9** contains the **Table of Logical Equivalences**.
- The maximum points possible = 44 points.
- Read all questions carefully and be sure to follow the instructions for the individual problems. You may ask for reasonable clarifications.
- Unless indicated otherwise, you must show your work in order to earn full marks – your solution must be complete, correct, and contain all relevant details.
- For additional work space, you may use the backs of pages. **Do not use any of your own scrap paper**.
- You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read and understood, and will comply with the above instructions.**

Family Name:	Student Number:
First Name:	†Signature:

SOLUTIONS

Do not write in this table.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Total
Maximum points	2 pts	6 pts	7 pts	6 pts	4 pts	4 pts	5 pts	5 pts	5 pts	44 points
Marks obtained										

Q1. [2 points] Fill in the following truth table.

P	Q	$P \wedge Q$	$P \vee Q$	$P \oplus Q$	$P \rightarrow Q$	$P \leftrightarrow Q$	$\neg P$
T	**T**	T	T	F	T	T	F
T	**F**	F	T	T	F	F	F
F	**T**	F	T	T	T	F	T
F	**F**	F	F	F	T	T	T

Q2. [6 points] For each proposition below, circle the appropriate response to indicate whether the given compound proposition is a tautology, contradiction, or contingency, or whether you do not have sufficient information to determine its type. No explanation is needed for this question.

The variables Q, R, and S represent *mystery* compound propositions of the following types:

Q is a **tautology**, R is a **contradiction**, and S is a **contingency**.

i. $Q \vee S$ $\equiv T \vee S \equiv T$ — tautology — contradiction — contingency — unable to determine

ii. $(Q \vee R) \rightarrow S$ — tautology — contradiction — contingency — unable to determine

$\equiv (T \vee F) \rightarrow S \equiv T \rightarrow S \equiv F \vee S \equiv S$

iii. $R \rightarrow (S \vee \neg Q)$ — tautology — contradiction — contingency — unable to determine

$\equiv F \rightarrow (S \vee \neg Q)$

$\equiv T$

iv. $\neg S$ — tautology — contradiction — contingency — unable to determine

$(T \rightarrow F) \rightarrow S \equiv F \rightarrow S \equiv T$

v. $(Q \rightarrow R) \rightarrow S$ — tautology — contradiction — contingency — unable to determine

$T \rightarrow (F \wedge S) \equiv F$

vi. $Q \rightarrow (R \wedge S)$ — tautology — contradiction — contingency — unable to determine

Q3a. [4 points] Translate each of the following statements into propositional logic using the propositional variables A, H, L, defined below, appropriate logical connectives, and parentheses when necessary:

A: "The shark attacks Lee." **H**: "The shark is hungry." **L**: "Lee is a knight."

*You may assume Lee is an inhabitant of the Island of Knights & Knaves.

P_1: "Lee is a knave only if the shark is not hungry."

Translation of P_1 into propositional logic: $\neg L \to \neg H$

P_2: "The shark is hungry but the shark does not attack Lee."

Translation of P_2 into propositional logic: $H \land \neg A$

P_3: "The shark attacks Lee unless Lee is a knight."

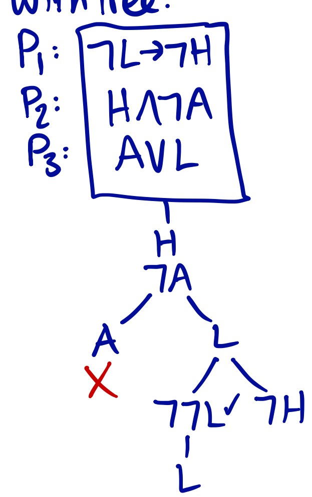
Translation of P_3 into propositional logic: $A \lor L$

P_4: "A necessary condition for the shark to attack Lee is that the shark is hungry."

Translation of P_4 into propositional logic: $A \to H$

Q3b. [3 points] Using the method of your choice, determine whether the set $\{P_1, P_2, P_3\}$ of the first three propositions from part (a) is **consistent**. If this set is **consistent**, give **all** truth assignments that justify your answer. Show your work and briefly explain your conclusions.

with tree:



with table

A	H	L	$A \lor L$	$\neg L \to \neg H$	$H \land \neg A$
T	T	T	T	T	F
T	T	F	T	F	F
T	F	T	T	T	F
T	F	F	T	T	F
F	T	T	T	T	T
F	T	F	F	F	T
F	F	T	T	T	F
F	F	F	F	T	F

yes the set $\{P_1, P_2, P_3\}$ is consistent

All three propositions are true when $A = F, H = F, L = T$

Q4. [6 points] For each of the following arguments, determine whether or not it is a **valid** argument. Show your work! If you claim the argument is not valid, give **all** counterexamples.

Q4a $W \wedge X$

$\underline{W \rightarrow X}$

$\therefore W \vee X$

Is this argument valid? *Circle:* **YES** **NO**

If you circled, **NO**, give all counterexamples!

$W \wedge X$ ✓
$W \rightarrow X$
$\neg(W \vee X)$ ✓
|
W
X
|
$\neg W$
$\neg X$
X

Q4b $\underline{W}$

$\therefore \neg W \rightarrow X$

Is this argument valid? *Circle:* **YES** **NO**

If you circled, **NO**, give all counterexamples!

W ✓
$\neg(\neg W \rightarrow X)$
|
$\neg W$
$\neg X$
X

Q4c $\neg(W \rightarrow X)$

$\underline{W \vee Y}$

$\therefore \neg Y$

Is this argument valid? *Circle:* **YES** **NO**

If you circled, **NO**, give all counterexamples!

$\neg(W \rightarrow X)$ ✓
$W \vee Y$
$\neg\neg Y$ ✓
|
Y
|
W
$\neg X$
/ \
W Y

counterexamples:

- when $W=T$, $X=F$, $Y=T$, both premises are true but the conclusion is false

Q5. [4 points] On the Island of Knights & Knaves, you meet an inhabitant Lee who is either a knight or a knave. Lee says: "If the shark attacks me, then I am a knave."

What (if anything) can you conclude? Be as specific as possible about Lee's type and whether the shark attacks Lee. Clearly define any variables that enter into your solution.

Define: L : Lee is a Knight A: The shark attacks

says: $A \to \neg L$

L	A	$A \to \neg L$	
T	T	F	↯
T	F	T	← This is the only possible solution.
F	T	T	↯
F	F	T	↯

∴ Lee is a Knight

The shark does not attack Lee.

Q6. [4 points] Using the laws in the **Table of Logical Equivalences** (on Page ??), show that

$$(A \vee B) \to C \equiv (A \to C) \wedge (B \to C)$$

Justify each step by giving the name or number of the corresponding equivalence on Page ??. *Do not skip steps. Do not combine more than two equivalences into a single step. Do not omit necessary parentheses.*

$(A \vee B) \to C \equiv \neg(A \vee B) \vee C$ Implication Law

$\equiv (\neg A \wedge \neg B) \vee C$ De Morgan's Law

$\equiv C \vee (\neg A \wedge \neg B)$ Commutative Law

$\equiv (C \vee \neg A) \wedge (C \vee \neg B)$ Distributive Law

$\equiv (\neg A \vee C) \wedge (\neg B \vee C)$ Commutative Law (twice)

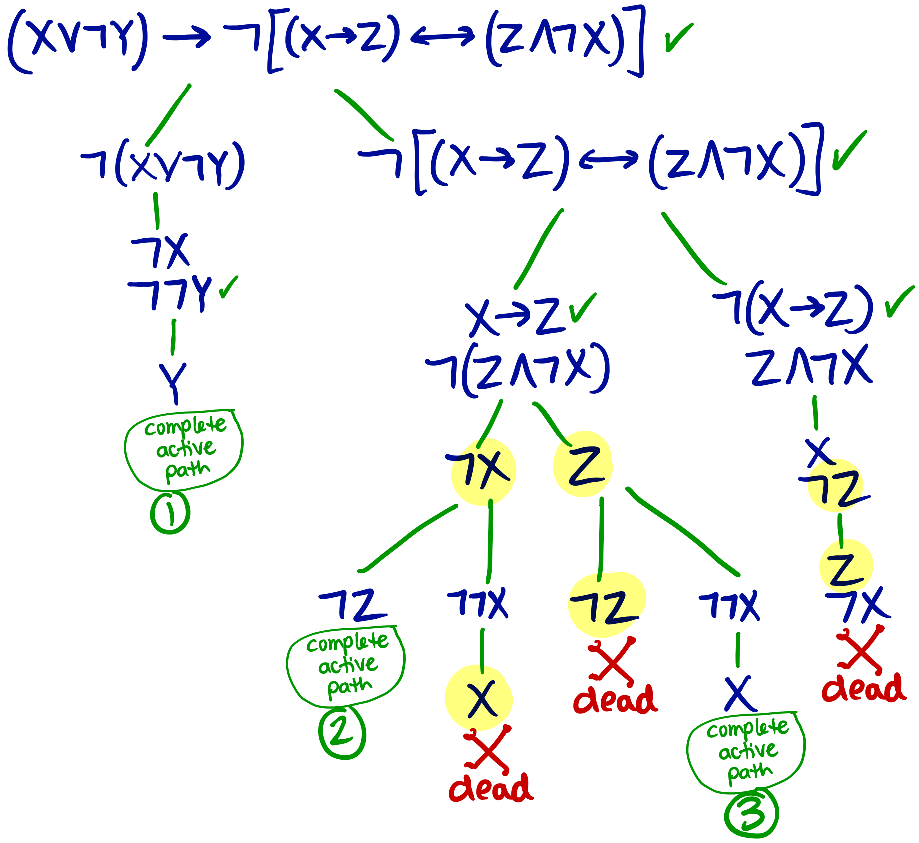
$\equiv (A \to C) \wedge (B \to C)$ Implication Law (twice)

Q7. [5 points] Using an appropriate truth tree, determine whether or not the compound proposition P, defined as follows, is a **contradiction**.

$$P: \quad \big(X \vee \neg Y\big) \to \neg\Big[\big(X \to Z\big) \leftrightarrow \big(Z \wedge \neg X\big)\Big]$$

Grow a **complete** truth tree. Clearly label each path as either "active" or "dead". Apply the branching rules to the propositions as written (i.e. do not rewrite a proposition using logical equivalences before branching).

Complete Truth Tree:



Is P a **contradiction**? *Circle:* **YES** **NO**

Briefly explain making reference to your tree and its root.

Since there is at least one complete active path, the root P can be true (hence is not a contradiction)

Give a disjunctive normal form (DNF) for P:

$(\neg X \wedge Y) \vee (\neg X \wedge \neg Z) \vee (X \wedge Z)$

Q8. [5 points] Let d and k be integers. Recall that $d \mid k$ is notation for "d divides k", which means that k is an integer multiple of d, and $d \nmid k$ is notation for "d does not divide k".
The goal of this question is to prove the following theorem:

Theorem 1. Let n be an integer. If $4 \nmid n$, then $4 \nmid (n^2+1)$.

☐ For each variable that appears in your proof, you must state what that variable represents.

☐ For each step in your proof, it must be made clear to the reader whether it is an assumption, something you are about to prove, or something that follows from a previous step or a definition.

hint: You may wish to consider the possible remainders of n divided by 4 and use a proof by cases.

Proof of Theorem 1.

Let n be an integer.

Assume $4 \nmid n$. Then $n = 4k+r$ for some integers k, r such that $0 < r < 4$

Case 1 Assume $n = 4k+1$, for some $k \in \mathbb{Z}$. Then

$n^2+1 = (4k+1)^2+1 = 16k^2+8k+2 = 4(\underbrace{4k^2+2k}) + 2 = 4j+2$ where $j = 4k^2+2k$. since $k \in \mathbb{Z}$, $j \in \mathbb{Z}$

$\therefore$ n^2+1 has remainder 2 $\Rightarrow 4 \nmid n^2+1$ in Case 1.

Case 2 Assume $n = 4k+2$, for some $k \in \mathbb{Z}$. Then

$n^2+1 = (4k+2)^2+1 = 16k^2+16k+5 = 4(\underbrace{4k^2+4k+1}) + 1 = 4j+1$ where $j = 4k^2+4k+1$. since $k \in \mathbb{Z}$, $j \in \mathbb{Z}$

$\therefore$ n^2+1 has remainder 1 $\Rightarrow 4 \nmid n^2+1$ in Case 2.

Case 3 Assume $n = 4k+3$, for some $k \in \mathbb{Z}$. Then

$n^2+1 = (4k+3)^2+1 = 16k^2+24k+10 = 4(4k^2+6k+2) + 2 = 4j+2$ where $j = 4k^2+6k+2$. since $k \in \mathbb{Z}$, $j \in \mathbb{Z}$

$\therefore$ n^2+1 has remainder 2 $\Rightarrow 4 \nmid n^2+1$ in Case 1.

In all possible cases, we find that $4 \nmid n^2+1$ $\therefore$ $4 \nmid n \rightarrow 4 \nmid n^2+1$ is true ∎

Q9. [5 points] For this question, you will give an **indirect proof** of the following theorem:

Theorem 2. Let x be a positive real number. If x is irrational, then $\sqrt{2x+3}$ is irrational.

Start by writing Theorem 2 in its **contrapositive** form:

Contrapositive of Theorem 2 (in English): Let x be a positive real number. If $\sqrt{2x+3}$ is rational, then x is rational.

Complete the definition:

A real number x is a **rational number** if...
$x = \frac{m}{n}$ for some integers m, n such that $n \neq 0$.

Now give an **indirect proof** of Theorem 2.

- ☐ For each variable that appears in your proof, you must state what that variable represents.
- ☐ For each step in your proof, it must be made clear to the reader whether it is an assumption, something you are about to prove, or something that follows from a previous step or a definition.

Indirect Proof of Theorem 2.

Let x be a positive real number.

Assume $\sqrt{2x+3}$ is rational.

Then $\sqrt{2x+3} = \frac{m}{n}$ for some integers m, n such that $n \neq 0$.

$\Rightarrow 2x+3 = \frac{m^2}{n^2}$

$\Rightarrow x = \left(\frac{m^2}{n^2} - 3\right)\left(\frac{1}{2}\right) = \frac{2m^2 - 6n^2}{2n^2}$

Since $m, n \in \mathbb{Z}$, this numerator is an integer.
Since 2, n are nonzero intgers, this denom. is a nonzero integer

$\therefore$ x is rational (by definition of rational).

	Equivalence	Name
1.	$P \to Q \equiv \neg P \vee Q$	Implication Law
2. 3.	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$ $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	Biconditional Laws
4. 5.	$P \vee \neg P \equiv \mathbf{T}$ $P \wedge \neg P \equiv \mathbf{F}$	Negation Laws
6. 7.	$P \vee \mathbf{F} \equiv P$ $P \wedge \mathbf{T} \equiv P$	Identity Laws
8. 9.	$P \vee \mathbf{T} \equiv \mathbf{T}$ $P \wedge \mathbf{F} \equiv \mathbf{F}$	Domination Laws
10. 11.	$P \vee P \equiv P$ $P \wedge P \equiv P$	Idempotent Laws
12.	$\neg\neg P \equiv P$	Double Negation Law
13. 14.	$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative Laws
15. 16.	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative Laws
17. 18.	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	Distributive Laws
19. 20.	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	De Morgan's Laws
21. 21.	$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$	Absorption Laws