Version A SOLUTIONS

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Calculus II

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MAT1322E – Test 2 – Monday, March 4, 2019

- Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:

◁ This is a 75-minute **closed-book** test. No notes. No calculators.

○ The exam consists of 6 questions on 6 pages (including this cover page).

◇ Each question is worth 2 points.

▷ maximum points possible = 12 points.

◆ Read all questions carefully and be sure to follow the instructions for the individual problems.

☐ All questions are ***long-answer. To receive full marks, your solution must be correct, complete, and show all relevant details.**

*Some pages include a multiple-choice question. You will **not** earn any points for circling the correct response without having properly justified your choice.

◀ You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.

▲ For additional work space, you may use the backs of pages.
Do not use any of your own scrap paper.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read, understood, and will comply with the above instructions.**

Family Name:	Student Number:
First Name:	†Signature:

1. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \ln(3n^2 - 2) - \ln(n^2 + 1)$.

Circle the most appropriate response regarding this sequence. Show your work!

A. converges to 3 **B.** diverges to $-\infty$ **C.** converges to $\ln(3)$ **D.** converges to $\ln(5)$

E. converges to e^5 **F.** diverges to ∞ **G.** converges to e^3 **H.** none of the previous answers

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln(3n^2-2) - \ln(n^2+1)$$
$$= \lim_{n\to\infty} \ln\left(\frac{3n^2-2}{n^2+1}\right)$$
$$= \ln\left(\lim_{n\to\infty} \frac{3n^2-2}{n^2+1}\right)$$
$$= \ln(3)$$

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2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ whose first few terms are

$$a_1 = \frac{1}{3}, \quad a_2 = -\frac{2}{9}, \quad a_3 = \frac{4}{27}, \quad a_4 = -\frac{8}{81}, \ldots$$

Find an expression for a_n and compute the sum S of the series $\sum_{n=1}^{\infty} a_n$. Show your work!

for $n \geq 1$, $a_n = \frac{(-2)^{n-1}}{3^n} = \frac{(-1)^{n-1}2^{n-1}}{3^n}$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{-2}{3}\right)^{n-1}$ is a geometric series with first term $a = \frac{1}{3}$ and common ratio $r = -\frac{2}{3}$ since $|r| = \left|-\frac{2}{3}\right| < 1$

it converges to $\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{\frac{1}{3}}{1+\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{5}{3}} = \frac{1}{3}\cdot\frac{3}{5} = \frac{1}{5}$

Circle the most appropriate response:

A. $a_n = \frac{(-1)^{n-1}2^n}{3^n}$ and $S = \frac{2}{5}$

B. $a_n = \frac{(-1)^n 2^{n-1}}{3^n}$ and $S = \frac{1}{5}$

C. $a_n = \frac{-2^n}{3^n}$ and $S = \frac{1}{5}$

D. $a_n = \frac{(-1)^{n-1}2^{n-1}}{3^n}$ and $S = 1$

E. $a_n = \frac{(-1)^{n-1}2^{n-1}}{3^n}$ and $S = \frac{1}{5}$

F. none of the previous answers

3. Which of the following series converge? Show your work and justify your answers!

I. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k(3k+1)}{k!}$ Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(3(n+1)+1)/(n+1)!}{2^n(3n+1)/n!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{2^{n+1}\cdot(3n+4)\cdot n!}{2^n(3n+1)\cdot(n+1)!}\right|$$
$$= \lim_{n\to\infty}\frac{2(3n+4)}{(3n+1)(n+1)} = \lim_{n\to\infty}\frac{6n+8}{3n^2+4n+1} = 0 < 1$$

$\therefore$ this series is absolutely convergent by virtue of Ratio Test.

II. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{n+2}$

Test for Divergence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(-1)^n n}{n+2} = \left(\lim_{n\to\infty}(-1)^n\right)\left(\lim_{n\to\infty}\frac{n}{n+1}\right) = \left(\lim_{n\to\infty}(-1)^n\right)(1) = \lim_{n\to\infty}(-1)^n \text{ DNE}$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, this series diverges.

III. $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+n}{3n^3-2}$ Comparison Test: for all $n\geq 1$, $\dfrac{2n^2+n}{3n^3-2} \geqslant \dfrac{2n^2}{3n^3} = \dfrac{2}{3n} \geqslant 0$

since $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3n} = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges $(p=1)$ (harmonic series), $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+n}{3n^3-2}$ must also diverge.

Limit Comparison Test:

for all $n\geq 1$, $a_n = \dfrac{2n^2+n}{3n^3-2} > 0$ and $b_n = \dfrac{1}{n} > 0$ $\displaystyle\lim_{n\to\infty}\frac{\frac{2n^2+n}{3n^3-2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{2n^3+n^2}{3n^3-2} = \frac{2}{3} > 0$

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ exists and equals a positive constant, Σa_n and Σb_n behave the same way

$\displaystyle\sum b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ is divergent $\therefore$ $\displaystyle\sum a_n = \sum_{n=1}^{\infty}\frac{2n^2+n}{3n^3-2}$ is also divergent.

Circle the most appropriate response:

A. I, II, and III all converge. **B.** only II and III converge **C.** none of these series converge

D. only I and III converge **E.** only I converges **F.** only II converges **G.** only III converges

H. none of the above

4. Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{2^n \cos(n)}{4^n - n^2}$.

Of the following words, circle *all* those words (if any) which accurately describe this series.

~~GEOMETRIC~~ ~~ALTERNATING~~ ABSOLUTELY ~~CONDITIONALLY~~ CONVERGENT ~~DIVERGENT~~

For each word you circled, you must briefly justify why it describes the series in the space below.

$$a_n = \frac{(-1)^n \cos(n)}{4^n - n^2}$$

- since $\cos(n)$'s sign changes in a non-alternating pattern, this is not an alternating series
- It's not a geometric series either because there is not a common ratio between consecutive terms

Consider $\sum_{n=1}^{\infty} |a_n|$ instead.

$$0 \leq \sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot 2^n \cdot \cos(n)}{4^n - n^2} \right| = \sum_{n=1}^{\infty} \frac{2^n |\cos(n)|}{4^n - n^2} \leqslant \sum_{n=1}^{\infty} \frac{2^n}{4^n - n^2}$$

use Limit Comparison Test on this series with $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{2^n}{4^n - n^2}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{4^n}{4^n - n^2} = \lim_{n\to\infty} \frac{4^n \cdot (1)}{4^n(1 - \underbrace{\tfrac{n^2}{4^n}}_{\to 0 \text{ as } n\to\infty})} = 1$$

Since limit exists and equals a positive constant $(c=1)$, We know that $\sum_{n=1}^{\infty} \frac{2^n}{4^n - n^2}$ behaves like $\sum_{n=1}^{\infty} \frac{1}{2^n}$ by virtue of the Limit Comparison Test.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with common ratio $r = \frac{1}{2}$ hence it's convergent. $\therefore \sum_{n=1}^{\infty} \frac{2^n}{4^n - n^2}$ is convergent.

By the Comparison Test, we see that $\sum |a_n|$ is convergent.

$\therefore \sum a_n$ is absolutely convergent (hence convergent)

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

We want to estimate its sum with the sum of its first n terms. How many terms to we need to add so that the error in our estimate R_n is guaranteed to be less than 0.01 ?

A. 7 **B.** 8 **C.** 15 **D.** 14 **E.** ∞ **F.** 11 **G.** 10

H. none of the above

Show your work!

Let $f(x) = \frac{1}{x^3}$. Then f is positive, continuous and decreasing for all $x > 0$
$\Rightarrow$ Integral Test & Remainder Estimate Theorem apply.

$$R_n \leq \int_n^\infty \frac{1}{x^3}\,dx$$

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{T\to\infty} \int_n^T x^{-3}\,dx$$

$$= \lim_{T\to\infty} -\frac{1}{2}x^{-2}\Big|_n^T$$

$$= \lim_{T\to\infty} -\frac{1}{2T^2} - \left(-\frac{1}{2n^2}\right)$$

$$= 0 + \frac{1}{2n^2}$$

To have $R_n < 0.01$ it suffices to find n such that $\int_n^\infty \frac{1}{x^3}\,dx < 0.01$

$\Rightarrow$ Solve for n in the inequality $\frac{1}{2n^2} < 0.01$

$$\Rightarrow \frac{1}{2n^2} < \frac{1}{100}$$

$$\Rightarrow \frac{100}{2} < n^2$$

$$\Rightarrow 50 < n^2$$

$$\Rightarrow \sqrt{50} < n$$

Since $7^2 = 49$ and $8^2 = 64$, it follows that we need $n \geq 8$
(since we need $n > \sqrt{50} > \sqrt{49} = 7$ and n must be an integer)

6. Consider the series $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \ldots$

If we estimate its sum with the sum of its first $n = 50$ terms, how big could the error be?

The error could be at most... (circle the best answer)

A. $\frac{1}{50}$ **B.** $\frac{2}{103}$ **C.** $\frac{1}{103}$ **D.** $\frac{2}{101}$ **E.** 0 **F.** $\frac{2}{101^2}$

G. none of the above

Fully justify your answer below.

$$\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 2}{2n+1}$$

So $a_n = \frac{(-1)^{n-1}\, 2}{2n+1}$ for $n \geq 1$

$b_n = |a_n| = \frac{2}{2n+1}$

This is an alternating series

i) $b_{n+1} = \frac{2}{2(n+1)+1} = \frac{2}{2n+3} < \frac{2}{2n+1} = b_n$ ✓

ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{2n+1} = 0$ ✓

$\therefore$ this series is convergent by virtue of the AST.

Moreover, $|R_n| \leq b_{n+1}$ by ASET.

If we use S_{50} to estimate this series' sum, then $|R_{50}| \leq b_{51} = \frac{2}{2(51)+1} = \frac{2}{103}$

Do not write in this table. page 6 of 6

Question	Q1	Q2	Q3	Q4	Q5	Q6	Total
Maximum points	2 pts	2 pts	2 pts	2 pts	2 pts	2 pts	12 points
Marks obtained							

(end of Test 2)