Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Calculus II

Instructor: Elizabeth Maltais

MAT1322E – Test 1 – Wednesday, January 23, 2019

- ◁ Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:
- ◁ This is a 75-minute **closed-book** test. No notes. No calculators.
- ◁ The exam consists of 6 questions on 7 pages (including this cover page).
- ▷ Each question is worth 2 points.
- ▷ maximum points possible = 12 points.
- ◁ Read all questions carefully and be sure to follow the instructions for the individual problems.
- ◁ All questions are *long-answer. **To receive full marks, your solution must be correct, complete, and show all relevant details.**

  *Some pages end with a multiple-choice question based on that page's long-answer question. You will **not** earn any points for circling the correct response without having properly justified your choice in the long-answer section.
- ◁ You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.
- ◁ For additional work space, you may use the backs of pages.
  Do not use any of your own scrap paper.

  Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

  † **By signing below, you acknowledge that you have read, understood, and will comply with the above instructions.**

Family Name:	Student Number:
First Name:	†Signature:

SOLUTIONS

Do not write in this table.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Total
Maximum points	2 pts	2 pts	2 pts	2 pts	2 pts	2 pts	12 points
Marks obtained							

1a. Write down the formula for the average value of a function $f(x)$ over the interval $[a, b]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

1b. Find the average value of the function $f(x) = \dfrac{x^2}{\sqrt{x^3+1}}$ on the interval $[0, 2]$.

$$f_{ave} = \frac{1}{2-0}\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx$$

$u = x^3+1$

$\frac{du}{dx} = 3x^2 \implies dx = \frac{du}{3x^2}$

$x = 2 \implies u = 2^3+1 = 9$

$x = 0 \implies u = 0^3+1 = 1$

$$= \frac{1}{2}\int_{u=1}^{u=9} \frac{\cancel{x^2}}{\sqrt{u}}\frac{du}{3\cancel{x^2}}$$

$$= \frac{1}{6}\int_1^9 u^{-1/2}\,du$$

$$= \frac{1}{6}\left[2u^{1/2}\right]_1^9$$

$$= \frac{1}{6}\left[2\sqrt{9} - 2\sqrt{1}\right]$$

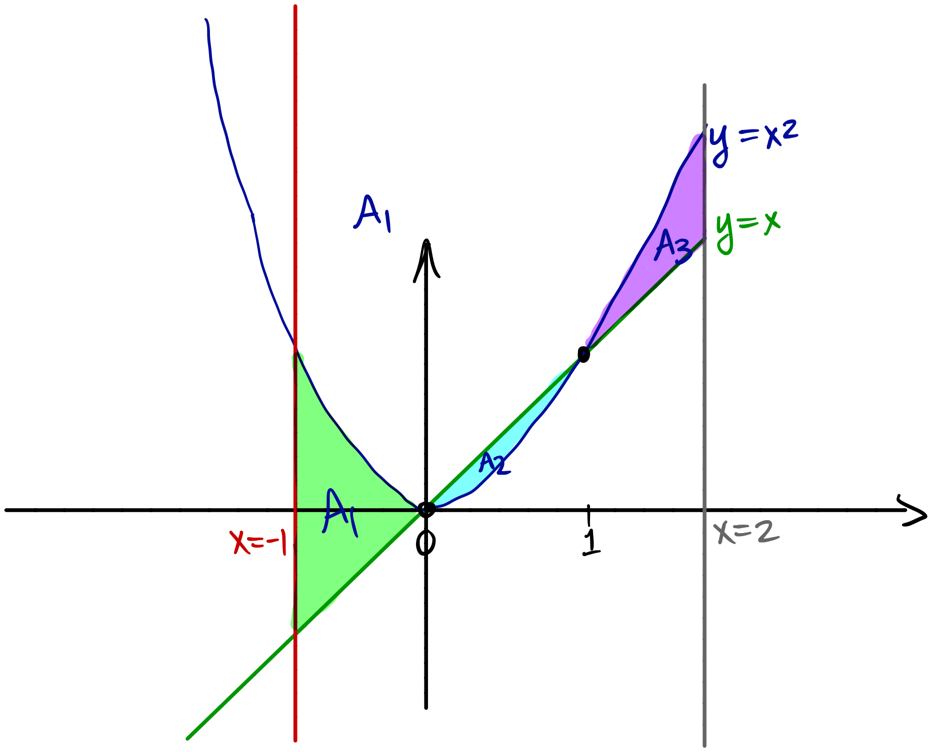
$$= \frac{1}{3}(3-1)$$

$$= \frac{2}{3}$$

1c. What is its average value? Circle the best response:

A. $\frac{4}{3}$ $\boxed{\textbf{B. } \frac{2}{3}}$ **C.** $\frac{8}{3}$ **D.** $-\frac{10}{3}$ **E.** 2 **F.** 0 **G.** none of the previous answers

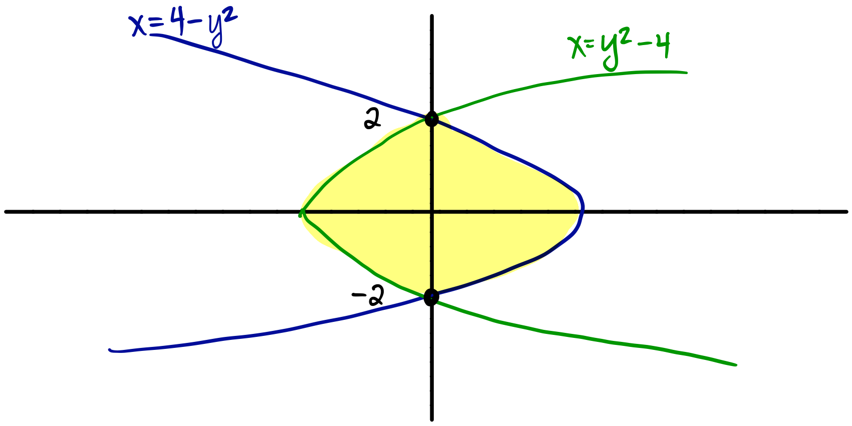
2a. Sketch the region(s) bounded by the curves $y = x^2$ and $y = x$ and the lines $x = -1$ and $x = 2$.



2b. Write down the integral(s) needed to find the total area of the above region(s).
Note: You do **not** need to evaluate your integral(s).

$$\text{Total Area} = \underbrace{\int_{-1}^{0} x^2 - x \, dx}_{A_1} + \underbrace{\int_{0}^{1} x - x^2 \, dx}_{A_2} + \underbrace{\int_{1}^{2} x^2 - x \, dx}_{A_3}$$

3a. Sketch the region enclosed by the curves $x = 4 - y^2$ and $x = y^2 - 4$.



POI

$4-y^2 = y^2-4$

$\Rightarrow 2y^2-8=0$

$y^2-4=0$

$(y-2)(y+2)=0$

$y=2 \qquad y=-2$

3b. Write down the integral needed to find the total area of the above region and fully evaluate it.

$$A = \int_{y=-2}^{y=2} \left(4-y^2-(y^2-4)\right) dy$$

$$= \int_{-2}^{2} 8-2y^2 \, dy$$

$$= \left[8y - \frac{2y^3}{3}\right]_{-2}^{2}$$

$$= \left(8(2) - \frac{2\cdot 2^3}{3}\right) - \left(8(-2) - \frac{2(-2)^3}{3}\right)$$

$$= 16 - \frac{16}{3} - \left(-16 + \frac{16}{3}\right)$$

$$= 32 - \frac{32}{3}$$

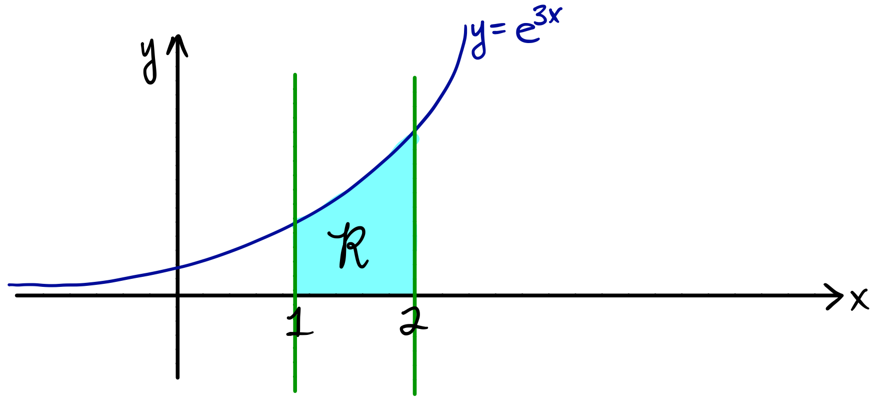
$$= \frac{96-32}{3}$$

$$= \frac{64}{3}$$

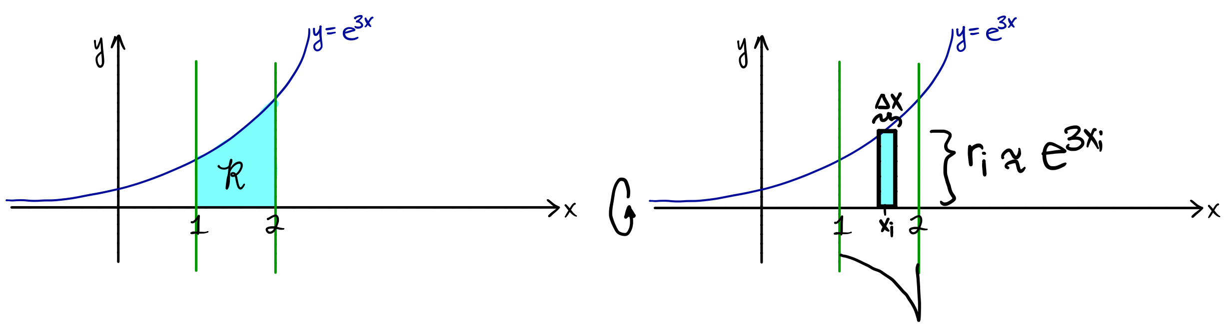
3c. What is the total area? Circle the best response:

A. 21 **B.** $\frac{128}{3}$ **C.** $-\frac{37}{3}$ **D.** 0 **E.** $-\frac{64}{3}$ $\boxed{\textbf{F.}\ \frac{64}{3}}$ **G.** none of the previous answers

4a. Sketch the region $\mathcal{R}$ enclosed by the curve $y = e^{3x}$ and the x-axis and the lines $x = 1$ and $x = 2$.



4b. Let $\mathcal{S}$ be the solid obtained by rotating $\mathcal{R}$ about the x-axis. Find the exact volume of $\mathcal{S}$. Show your work!



approx. volume of ith slice $\quad V_i \approx \pi r_i^2 \Delta x \approx \pi \left(e^{3x_i}\right)^2 \Delta x$

approx. total volume of S $\quad V \approx \sum_{i=1}^{n} \pi \left(e^{3x_i}\right)^2 \Delta x$

exact volume of S: $V = \int_{x=1}^{x=2} \pi \left(e^{3x}\right)^2 dx$

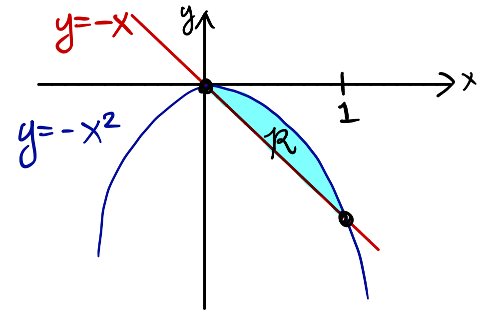
$$= \int_1^2 \pi e^{6x} dx$$

$$= \pi \left[\frac{1}{6} e^{6x}\right]_1^2$$

$$= \frac{\pi}{6}\left[e^{6(2)} - e^{6(1)}\right]$$

$$= \frac{\pi}{6}\left[e^{12} - e^{6}\right]$$

5a. Sketch the region $\mathcal{R}$ bounded by the curves $y = -x^2$ and $y = -x$.



5b. Find the centroid (centre of mass) of $\mathcal{R}$. Show your work!

$$A = \int_0^1 -x^2-(-x)\,dx = \int_0^1 -x^2+x\,dx = \left[-\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1$$

$$= \left(-\frac{1^3}{3}+\frac{1^2}{2}\right)-\left(-\frac{0^3}{3}+\frac{0^2}{2}\right) = -\frac{1}{3}+\frac{1}{2} = -\frac{2}{6}+\frac{3}{6} = \frac{1}{6}$$

$$\bar{x} = \frac{1}{A}\int_0^1 x\left(-x^2-(-x)\right)dx = \frac{1}{\frac{1}{6}}\int_0^1 -x^3+x^2\,dx = 6\left[-\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1$$

$$=6\left[\left(-\frac{1^4}{4}+\frac{1^3}{3}\right)-\left(-\frac{0^4}{4}+\frac{0^3}{3}\right)\right] = 6\left[-\frac{1}{4}+\frac{1}{3}\right] = 6\left[-\frac{3}{12}+\frac{4}{12}\right] = 6\left[\frac{1}{12}\right] = \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left\{(-x^2)^2-(-x)^2\right\}dx = \left(\frac{1}{\frac{1}{6}}\right)\left(\frac{1}{2}\right)\int_0^1 x^4-x^2\,dx = 3\left[\frac{x^5}{5}-\frac{x^3}{3}\right]$$

$$=3\left[\left(\frac{1^5}{5}-\frac{1^3}{3}\right)-\left(\frac{0^5}{5}-\frac{0^3}{3}\right)\right] = 3\left[\frac{1}{5}-\frac{1}{3}\right] = 3\left[\frac{3}{15}-\frac{5}{15}\right] = 3\left(\frac{-2}{15}\right) = \frac{-2}{5}$$

5c. What is its centroid? Circle the best response:

A. $(-\frac{1}{2}, \frac{2}{5})$ **B.** $(\frac{1}{2}, -\frac{2}{5})$ **C.** $(-\frac{1}{2}, -\frac{2}{5})$ **D.** $(\frac{1}{2}, -\frac{1}{2})$

E. $(-\frac{1}{2}, \frac{1}{2})$ **F.** $(-\frac{1}{2}, -\frac{1}{2})$ **G.** none of the previous answers

6a. Is the integral $I = \int_{-2}^{7} \frac{6}{\sqrt[3]{x+1}}\,dx$ proper or improper? Briefly explain.

It is improper because the integrand is discontinuous when $x=-1$ which is on the interval of integration $[-2,7]$

6b. Using appropriate mathematical notation and methods, evaluate the integral:

$$I = \int_{-2}^{7} \frac{6}{\sqrt[3]{x+1}}\,dx$$

Show your work!

$$I = \int_{-2}^{-1} \frac{6}{\sqrt[3]{x+1}}\,dx + \int_{-1}^{7} \frac{6}{\sqrt[3]{x+1}}\,dx$$

$$= \lim_{t\to -1^-} \int_{-2}^{t} 6(x+1)^{-\frac{1}{3}}\,dx + \lim_{s\to -1^+} \int_{s}^{7} 6(x+1)^{-\frac{1}{3}}\,dx$$

$$= \lim_{t\to -1^-} \left[6\left(\tfrac{3}{2}\right)(x+1)^{2/3}\right]_{-2}^{t} + \lim_{s\to -1^+} \left[6\left(\tfrac{3}{2}\right)(x+1)^{\frac{2}{3}}\right]_{s}^{7}$$

$$= \lim_{t\to -1^-} \left[\underbrace{9(t+1)^{2/3}}_{\to 0} - 9(-2+1)^{2/3}\right] + \lim_{s\to -1^+} \left[9(7+1)^{2/3} - \underbrace{9(s+1)^{2/3}}_{\to 0}\right]$$

$$= 0 - 9(-1)^{2/3} + 9(8)^{\frac{2}{3}} - 0$$

$$= -9(1) + 9(2)^2$$

$$= -9 + 36 = 27$$

6c. Circle the answer that best describes the integral I.

A. proper, divergent, $I = 27$

B. proper, convergent, $I = 0$

C. improper, convergent, $I = 27$

D. improper, divergent

E. improper, convergent $I = -27$

F. improper, convergent $I = 0$

G. none of the above

(end of Test 1)