Discrete Structures I
COMP1805

Establishing Logical Equivalence

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in Logic, the Set of Possible Values is Finite (i.e., Truth Tables show Every Possible Evaluation)

thus

Expressions (and Corresponding Assertions) are Equivalent If and Only If Truth Tables are Identical

to Clarify this, Reconsider Arithmetic Expressions

$$2 \cdot x + 4 = 6 \cdot \frac{x+2}{3}$$

Without Formal Proof, it is Impossible to Show an Equivalence as it is Possible for x to be Any Number...

...but if x could only be a value between 1 and 10, you could Evaluate the Left and Right Expressions for Every Possible Value

Logical Equivalence Example

Prove (by Truth Table) the Following Equivalence

$$P \rightarrow Q \quad \equiv \quad \neg Q \rightarrow \neg P$$

P	*Q*
T	*T*
T	*F*
F	*T*
F	*F*

Logical Equivalence Example

Prove (by Truth Table) the Following Equivalence

$$P \rightarrow Q \quad \equiv \quad \neg Q \rightarrow \neg P$$

P	Q	$P \rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \rightarrow \neg P$
T	*T*	*T*	*F*	*F*	*T*
T	*F*	*F*	*F*	*T*	*F*
F	*T*	*T*	*T*	*F*	*T*
F	*F*	*T*	*T*	*T*	*T*

Logical Equivalence Example

Prove (by Truth Table) the Following Equivalence

$$\neg (P \wedge \neg Q) \equiv \neg P \vee Q$$

P	*Q*
T	*T*
T	*F*
F	*T*
F	*F*

Logical Equivalence Example

Prove (by Truth Table) the Following Equivalence

$$\neg(P \wedge \neg Q) \quad = \quad \neg P \vee Q$$

P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$	$\neg P$	$\neg P \vee Q$
T	*T*	*F*	*F*	*T*	*F*	*T*
T	*F*	*T*	*T*	*F*	*F*	*F*
F	*T*	*F*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*T*	*T*	*T*

Logical Equivalence Example

Rewrite $P \oplus Q$ using Only the $\neg$, $\wedge$, and $\vee$ Operators

Prove that your Solution is Correct (by Truth Table)

Logical Equivalence Example

Rewrite $P \oplus Q$ using Only the $\neg$, $\wedge$, and $\vee$ Operators

Prove that your Solution is Correct (by Truth Table)

$$P \oplus Q$$

can be translated into

Either *P* Or *Q*, But Not Both

can be translated into

$$(P \vee Q) \wedge \neg (P \wedge Q)$$

Logical Equivalence Example

Prove (by Truth Table) the Following Equivalence

$$P \oplus Q \;=\; (P \vee Q) \wedge \neg (P \wedge Q)$$

P	*Q*
T	*T*
T	*F*
F	*T*
F	*F*

Prove (by Truth Table) the Following Equivalence

$$P \oplus Q \equiv (P \vee Q) \wedge \neg (P \wedge Q)$$

P	Q	$P \oplus Q$	$P \vee Q$	$P \wedge Q$	$\neg (P \wedge Q)$	$(P \vee Q) \wedge \neg (P \wedge Q)$
T	*T*	*F*	*T*	*T*	*F*	*F*
T	*F*	*T*	*T*	*F*	*T*	*T*
F	*T*	*T*	*T*	*F*	*T*	*T*
F	*F*	*F*	*F*	*F*	*T*	*F*

Logical Equivalence Example

the Final Column in a Truth Table can be Used to Categorize an Expression

"Tautology" is an Expression that is Always True

"Contradiction" is an Expression that is Always False

Every Other Expression is a "Contingency"

Logical Equivalence Example

Demonstrate that
$P \to (P \vee Q)$ **is a Tautology**

Demonstrate that
$(P \leftrightarrow Q) \wedge (\neg P \wedge Q)$ **is a Contradiction**

Logical Equivalence Example

Demonstrate that

$P \rightarrow (P \vee Q)$ is a Tautology

P	Q	$P \vee Q$	$P \rightarrow (P \vee Q)$
T	*T*	*T*	*T*
T	*F*	*T*	*T*
F	*T*	*T*	*T*
F	*F*	*F*	*T*

Demonstrate that $(P \leftrightarrow Q) \wedge (\neg P \wedge Q)$ is a Contradiction

P	Q	$P \leftrightarrow Q$	$\neg P$	$\neg P \wedge Q$	$(P \leftrightarrow Q) \wedge (\neg P \wedge Q)$
T	*T*	*T*	*F*	*F*	*F*
T	*F*	*F*	*F*	*F*	*F*
F	*T*	*F*	*T*	*T*	*F*
F	*F*	*T*	*T*	*F*	*F*

Logical Equivalence Example

a **Fundamental Property** of **Equality** is **Transitivity**

(i.e., if $A = B$ and $B = C$ then $A = C$)

there are a Collection of Well Known Logical Equivalences that Allow one Expression to be Transformed into another

(n.b., this can be an Alternative to Truth Tables for Proving Equivalence)

Logical Equivalence Example

$$P \wedge T = P$$
$$Q \vee F = Q$$

Identity

$$P \wedge P = P$$
$$Q \vee Q = Q$$

Idempotent

$$P \vee T = T$$
$$Q \wedge F = F$$

Domination

Logical Equivalence Example

$$P \wedge Q = Q \wedge P$$
$$P \vee Q = Q \vee P$$

Commutative

$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R = P \vee (Q \vee R)$$

Associative

$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$
$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$

Distributive

Logical Equivalence Example

$$P \wedge \neg P = F$$
$$P \vee \neg P = T$$

Negation

$$P \wedge (P \vee Q) = P$$
$$P \vee (P \wedge Q) = P$$

Absorption

$$\neg(\neg P) = P$$

Double Negation

Logical Equivalence Example

$$P \rightarrow Q \equiv \neg P \vee Q$$

Implication Equivalence

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Biconditional Equivalence

$$\neg (P \wedge Q) \equiv \neg P \vee \neg Q$$
$$\neg (P \vee Q) \equiv \neg P \wedge \neg Q$$

De Morgan's Law

Logical Equivalence Example

Demonstrate the Logically Equivalence of

$\neg(A \vee (\neg A \wedge B))$ **and** $\neg A \wedge \neg B$

this can be Accomplished with Truth Tables

but it can Also be Accomplished with the Laws –

if you can Transform the First Expression into the Second Expression

Can you Match the Expression to be Transformed with Any of the Logical Equivalence Laws?

$$\neg(A \lor (\neg A \land B))$$

Logical Equivalence Example

$$\neg(A \lor (\neg A \land B))$$

this Expression Matches De Morgan's Law

$$\neg (P \lor Q) = \neg P \land \neg Q$$

if

P **is Used to Represent** ***A***

Q **is Used to Represent** $(\neg A \land B)$

Logical Equivalence Example

$\neg(A \vee (\neg A \wedge B))$

Logical Equivalence Example

$\neg(A \vee (\neg A \wedge B))$

$\neg A \wedge \neg(\neg A \wedge B)$ by the **De Morgan's Law**

Can you Match the Expression to be Transformed with Any of the Logical Equivalence Laws?

$$\neg A \land \neg(\neg A \land B)$$

Logical Equivalence Example

$$\neg A \land \neg(\neg A \land B)$$

this Subexpression Also Matches De Morgan's Law

$$\neg (P \land Q) = \neg P \lor \neg Q$$

if

P is Used to Represent $\neg A$

Q is Used to Represent B

Logical Equivalence Example

$\neg(A \vee (\neg A \wedge B))$

by the De Morgan's Law

$\neg A \wedge \neg(\neg A \wedge B)$

by the De Morgan's Law

$\neg A \wedge (\neg(\neg A) \vee \neg B))$

Logical Equivalence Example

$\neg(A \vee (\neg A \wedge B))$

by the De Morgan's Law

$\neg A \wedge \neg(\neg A \wedge B)$

by the De Morgan's Law

$\neg A \wedge (\neg(\neg A) \vee \neg B))$

by the Double Negation Law

$\neg A \wedge (A \vee \neg B)$

Logical Equivalence Example

$\neg(A \lor (\neg A \land B))$	
	by the De Morgan's Law
$\neg A \land \neg(\neg A \land B)$	
	by the De Morgan's Law
$\neg A \land (\neg(\neg A) \lor \neg B))$	
	by the Double Negation Law
$\neg A \land (A \lor \neg B)$	
	by the Distributive Law
$(\neg A \land A) \lor (\neg A \land \neg B)$	

Logical Equivalence Example

$\neg(A \lor (\neg A \land B))$	
	by the De Morgan's Law
$\neg A \land \neg(\neg A \land B)$	
	by the De Morgan's Law
$\neg A \land (\neg(\neg A) \lor \neg B))$	
	by the Double Negation Law
$\neg A \land (A \lor \neg B)$	
	by the Distributive Law
$(\neg A \land A) \lor (\neg A \land \neg B)$	
	by the Negation Law
$\text{False} \lor (\neg A \land \neg B)$	

Logical Equivalence Example

$\neg(A \lor (\neg A \land B))$

by the De Morgan's Law

$\neg A \land \neg(\neg A \land B)$

by the De Morgan's Law

$\neg A \land (\neg(\neg A) \lor \neg B))$

by the Double Negation Law

$\neg A \land (A \lor \neg B)$

by the Distributive Law

$(\neg A \land A) \lor (\neg A \land \neg B)$

by the Negation Law

$\text{False} \lor (\neg A \land \neg B)$

by the Commutative Law

$(\neg A \land \neg B) \lor \text{False}$

Logical Equivalence Example

$\neg(A \lor (\neg A \land B))$

by the De Morgan's Law

$\neg A \land \neg(\neg A \land B)$

by the De Morgan's Law

$\neg A \land (\neg(\neg A) \lor \neg B))$

by the Double Negation Law

$\neg A \land (A \lor \neg B)$

by the Distributive Law

$(\neg A \land A) \lor (\neg A \land \neg B)$

by the Negation Law

$\text{False} \lor (\neg A \land \neg B)$

by the Commutative Law

$(\neg A \land \neg B) \lor \text{False}$

by the Identity Law

$\neg A \land \neg B$

Logical Equivalence Example

Demonstrate (again) that $P \rightarrow (P \vee Q)$ is a Tautology (without using Truth Tables)

$P \rightarrow (P \vee Q)$

Demonstrate (again) that $P \to (P \lor Q)$ is a Tautology (without using Truth Tables)

$P \to (P \lor Q)$

by the Implication Equivalence

$\neg P \lor (P \lor Q)$

Demonstrate (again) that $P \rightarrow (P \vee Q)$ is a Tautology (without using Truth Tables)

$P \rightarrow (P \vee Q)$

by the Implication Equivalence

$\neg P \vee (P \vee Q)$

by the Associative Law

$(\neg P \vee P) \vee Q$

Logical Equivalence Example

Demonstrate (again) that $P \rightarrow (P \vee Q)$ is a Tautology (without using Truth Tables)

$P \rightarrow (P \vee Q)$

by the Implication Equivalence

$\neg P \vee (P \vee Q)$

by the Associative Law

$(\neg P \vee P) \vee Q$

by the Negation Law

$\text{True} \vee Q$

Demonstrate (again) that $P \to (P \lor Q)$ is a Tautology (without using Truth Tables)

$P \to (P \lor Q)$	
	by the Implication Equivalence
$\neg P \lor (P \lor Q)$	
	by the Associative Law
$(\neg P \lor P) \lor Q$	
	by the Negation Law
$\text{True} \lor Q$	
	by the Commutative Law
$Q \lor \text{True}$	

Logical Equivalence Example

Demonstrate (again) that $P \to (P \lor Q)$ is a Tautology (without using Truth Tables)

$P \to (P \lor Q)$	
	by the Implication Equivalence
$\neg P \lor (P \lor Q)$	
	by the Associative Law
$(\neg P \lor P) \lor Q$	
	by the Negation Law
$\text{True} \lor Q$	
	by the Commutative Law
$Q \lor \text{True}$	
	by the Domination Law
True	

a Logical Expression that is a Tautology is Always Demonstrably Equal to a Value of True...

and

a Logical Expression that is a Contradiction is Always Demonstrably Equal to a Value of False...

but

there are also Formats for Writing "Normalized" (i.e., "Standardized") Expressions in Logic

an Expression in "Disjunctive Normal Form" is a Disjunction of One or More Conjunctions

i.e., Conjunction ∨ Conjunction ∨ Conjunction...

e.g., $(A \land B) \lor (\neg B \land C) \lor (A \land \neg B \land C)$

the Only Operators Permitted are Negation (inside), Conjunction (inside), and Disjunction (outside)

in an Expression is "Full Disjunctive Normal Form", each variable Appears Exactly Once Per Conjunction

e.g., $(A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C)$

(assuming, of course, that the Only Variables are A, B, and C)

in Full Disjunctive Normal Form $R \to (H \land C)$ would be

$$(R \land H \land C) \lor (\neg R \land H \land C) \lor (\neg R \land H \land \neg C) \lor (\neg R \land \neg H \land C) \lor (\neg R \land \neg H \land \neg C)$$

since we Already Computed the Truth Table for this expression, this is Easily Confirmed...

in Full Disjunctive Normal Form $R \rightarrow (H \wedge C)$ would be

$$(R \wedge H \wedge C) \vee (\neg R \wedge H \wedge C) \vee (\neg R \wedge H \wedge \neg C) \vee (\neg R \wedge \neg H \wedge C) \vee (\neg R \wedge \neg H \wedge \neg C)$$

R	H	C	$H \wedge C$	$R \rightarrow (H \wedge C)$
T	*T*	*T*	*T*	*T*
T	*T*	*F*	*F*	*F*
T	*F*	*T*	*F*	*F*
T	*F*	*F*	*F*	*F*
F	*T*	*T*	*T*	*T*
F	*T*	*F*	*F*	*T*
F	*F*	*T*	*F*	*T*
F	*F*	*F*	*F*	*T*

Each of the Five Truth Table Rows that Evaluates to True Corresponds to Exactly One Conjunctive Clause

Disjunctive Normal Forms

$$R \rightarrow (H \wedge C)$$

How would you Convert this Expression into its Full Disjunctive Normal Form Without Completing the Truth Table?

Disjunctive Normal Forms

$R \rightarrow (H \wedge C)$	
$\neg R \vee (H \wedge C)$	**Implication Equivalence**
$\neg R \vee (True \wedge H \wedge C)$	**Identity**
$\neg R \vee ((R \vee \neg R) \wedge H \wedge C)$	**Negation**
$\neg R \vee ((R \wedge H \wedge C) \vee (\neg \mathrm{R} \wedge H \wedge C))$	**Distributivity**

…

How would you Expand $\neg R$ into the Remaining Clauses for the Full Disjunctive Normal Form?

Every Logic Expression can be Converted into One (and Only One) Full Disjunctive Form

although there are other real-world applications, you may find it most useful as the End "Goal" to Demonstrate that an Expression is a Contingency (and not, for example, a Tautology or Contingency)