University of Ottawa
MAT 1332 First Midterm Exam

Feb 14, 2018. Duration: 80 Minutes.
Instructor: Guy Beaulieau   Catalin Rada   Robert Smith?

Family Name:____________________

First Name:____________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to immediately leave the exam and academic fraud allegations will be filed, which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement:

  Signature:____________________
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed. All others will be confiscated.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- If you tear off any blank pages, they have to be handed in.
- Where it is possible to check your work, do so.
- Good luck!

Student number:____________________, Total marks: ___________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [5 points] Suppose that a tree trunk is 5 m high, and that its radius at height x m is $r(x) = \dfrac{e^{\frac{x}{2}}}{\sqrt{1+e^x}}$ m. Find its volume.

Solution: The integral that we need to compute (to get the volume) is: $I = \int_0^5 \pi(r(x))^2 dx = \pi \int_0^5 \frac{e^x}{1+e^x} dx$.

Use the following SUB: $u = 1 + e^x$, and notice that $\frac{du}{dx} = e^x$, or $du = e^x dx$. About limits of integration: when $x = 0$, one has that $u = 1 + e^0 = 2$; when $x = 5$, one has that $u = 1 + e^5$.

Our integral becomes now: $I = \pi \int_2^{1+e^5} \frac{du}{u} = \pi\{\ln(u)|_2^{1+e^5}\} = \pi\{\ln(1+e^5) - \ln(2)\}$.

Question 2. [3 points] Solve the separable differential equation

$$\frac{dy}{dx} = \frac{2e^{\sqrt{x}}}{y\sqrt{x}}$$

with **initial condition** $y(1) = 2\sqrt{e}$.

Circle one:

A) $y = \sqrt{8e^{\sqrt{x}} + 4e}$ B) $y = \sqrt{8e^{\sqrt{x}}}$ C) $y = -\sqrt{8e^{\sqrt{x}} - e}$

D) $y = -\sqrt{8e^{\sqrt{x}} - 4e}$ E) $y = \sqrt{8e^{\sqrt{x}} - 4e}$

ANSWER E)

Solution: This is a **Separable** Differential equation.

Separate: $ydy = \frac{2e^{\sqrt{x}}}{\sqrt{x}}dx$. **Integrate:** $\int ydy = \int \frac{2e^{\sqrt{x}}}{\sqrt{x}}dx$.

Compute as follows: (using the power rule and a SUB: $u = \sqrt{x}$, so $\frac{du}{dx} = \frac{1}{2}x^{\frac{-1}{2}}$)

$\frac{y^2}{2} = 2\int e^u 2du = 4\int e^u du = 4e^u + c = 4e^{\sqrt{x}} + c$, where c is a number.

The initial condition says: $2\sqrt{e} = y(1)$, so $4e = y(1)^2$, hence $2e = \frac{(y(1))^2}{2} = 4e^1 + c$, thus $c = 2e - 4e = -2e$.

From $\frac{y^2}{2} = 4e^{\sqrt{x}} - 2e$, one has $y = \pm\sqrt{8e^{\sqrt{x}} - 4e}$. Since the Initial Condition is Positive, it follows that the solution is ONLY: $y = \sqrt{8e^{\sqrt{x}} - 4e}$.

Question 3. [3 points] For the indefinite integral

$$\int \frac{2x^3 - 18x^2 + 39x + 1}{x^2 - 9x + 20}\, dx.$$

Which of the following terms does NOT appear in the solution?
Circle one:
A) x^2. B) $3\ln|x-4|$. C) $-4\ln|x+5|$. D) $-4\ln|x-5|$. E) $+c$
SOLUTION: By LONG DIVISION one has that $\frac{2x^3-18x^2+39x+1}{x^2-9x+20} = 2x + \frac{-x+1}{x^2-9x+20}$, which can be written as:

$2x + \frac{-x+1}{(x-4)(x-5)}$. For the second term we use Partial Fractions:

$\frac{-x+1}{(x-4)(x-5)} = \frac{A}{x-4} + \frac{B}{x-5}$, hence $-x+1 = A(x-5) + B(x-4) = x(A+B) - 5A - 4B$.

Hence $A + B = -1$, and $-5A - 4B = 1$,
Thus $A = -1 - B$ and $-5(-1-B) - 4B = 1$.
We get $B = -4$ and $A = 3$.

Our integral becomes: $x^2 + 3\ln|x-4| - 4\ln|x-5| + c$, c a number.

Question 4. [3 points] For the following improper integral, determine whether it converges, and determine its value if it does.

$$\int_2^{\infty} \frac{30}{4+8x^2}dx$$

A) It is divergent B) It is convergent and its value is 2.8 C) It is convergent and its value is 1.8

D) It is convergent and its value is 3.2 E) It is convergent and its value is 4.2

SOLUTION: By the very definition of an improper integral one has:

$I = \lim\limits_{t\to\infty} \int_2^t \frac{30}{4+8x^2}dx =$

$\frac{30}{4} \lim\limits_{t\to\infty} \int_2^t \frac{1}{1+2x^2}dx =$

$\frac{30}{4} \lim\limits_{t\to\infty} \int_2^t \frac{1}{1+(\sqrt{2}x)^2}dx.$

Now use a SUB: $u = \sqrt{2}x$, and notice that $\frac{du}{dx} = \sqrt{2}$, hence $\frac{du}{\sqrt{2}} = dx$.

Our integral becomes:

$I = \frac{30}{4} \lim\limits_{t\to\infty} \int_{2\sqrt{2}}^{t\sqrt{2}} \frac{1}{1+(u)^2}\frac{du}{\sqrt{2}} =$

$\frac{30}{4\sqrt{2}} \lim\limits_{t\to\infty} \arctan(u)|_{2\sqrt{2}}^{t\sqrt{2}} =$

$\frac{30}{4\sqrt{2}} \lim\limits_{t\to\infty} \{\arctan(t\sqrt{2}) - \arctan(2\sqrt{2})\} =$

$\frac{30}{4\sqrt{2}}\{\frac{\pi}{2} - \arctan(2\sqrt{2})\} \approx 1.8 \in \mathbf{R}$, thus it is convergent.

Question 5. [5 points] Consider the functions $y = f(x) = \frac{1}{x}$ and $y = g(x) = \frac{1}{x^2}$.

Find the area enclosed between the graphs of f, g and $x = 2$. Hint: First sketch the graphs of f and g, and find the intersection points (if any).

SOLUTION: To get the intersection points solve the equation $\frac{1}{x} = \frac{1}{x^2}$ as follows: $\frac{1}{x} - \frac{1}{x^2} = 0$, so $\frac{x-1}{x^2} = 0$, hence $x = 1$.

For $1 \leq x$ one has that $f(x) \geq g(x)$ (because $f(x) - g(x) = \frac{x-1}{x^2} \geq 0$), hence
$A = \int_1^2 \frac{1}{x} - \frac{1}{x^2} dx = \{\ln(x) + x^{-1}\}|_1^2 = \ln(2) + \frac{1}{2} - \ln(1) - 1 = \ln(2) - \frac{1}{2} \approx 0.193.$

Question 6. [6 points] Zombies have invaded campus! They recruit more of the undead at rate:

$$\frac{dz}{dt} = f(z) = k(z-1)(z+1)\ln(\frac{600}{z}),$$

where t is time and z is the number of zombies. Here k is a fixed positive constant.
a) Determine all biologically meaningful steady states (equilibrium points).
b) Determine the stability of each steady state in (a), using the derivative test.
c) Draw a phase-line diagram.
d) If 595 zombies are on campus initially, how many will there be eventually?

Solution: (a) We solve for z in $f(z) = 0$ as follows: $k(z-1)(z+1)\ln(\frac{600}{z}) = 0$, thus either $z-1 = 0$, $z+1 = 0$, or $\ln(\frac{600}{z}) = 0$; hence $z_1 = 1$, $z_2 = -1$ and $z_3 = 600$. The equilibrium points that are bio meaningful are: $z_1 = 1$ and $z_3 = 600$ since they are positive.

(b) We compute the derivative by product rule (since $f(z) = k((z^2-1))(\ln(600) - \ln(z)))$:

$$f'(z) = k\{(2z)(\ln(600) - \ln(z)) + (z^2-1)(-\frac{1}{z})\}$$

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Note that: $f'(1) = 2k\ln(600) > 0$, so $z_1 = 1$ is UNSTABLE. (Recall that $k > 0$.)

Note that: $f'(600) = k(600^2-1)\frac{-1}{600} < 0$, so $z_3 = 600$ is STABLE.

(c) Here is the phase line diagram:

(d) Since 595 is in between two equilibrium points (z_1 being unstable and z_3 being stable) and close to z_3, a solution starting at 595 converges to the stable equilibrium.

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Question 7. [5 points] Determine the average value of $f(x) = 2\ln(2x)$ over the range $1 \leq x \leq 3$.

SOLUTION: By its formula, the average value is $A = \frac{1}{3-1}\int_1^3 2\ln(2x)dx = \frac{1}{2} \times 2\int_1^3 1 \times \ln(2x)dx = \int_1^3 1 \times \ln(2x)dx$.

Use Integration by Parts: $u'(x) = 1$, and $v(x) = \ln(2x)$ imply that $u(x) = x$ and $v'(x) = \frac{2}{2x} = \frac{1}{x}$.

Thus $A = \{x\ln(2x)|_1^3 - \int_1^3 x\frac{1}{x}dx\} =$
$\{3\ln(6) - \ln(2) - \int_1^3 1dx\} =$
$\{3\ln(6) - \ln(2) - x|_1^3\} =$
$\{3\ln(6) - \ln(2) - 3 + 1\} =$
$\{3\ln(6) - \ln(2) - 2\} \approx 2.68.$

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