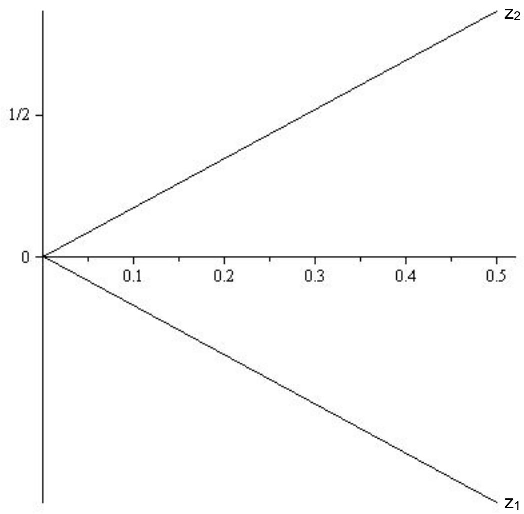
Question 1. Version A.

(a) Using the quadratic formula, we have
$z_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

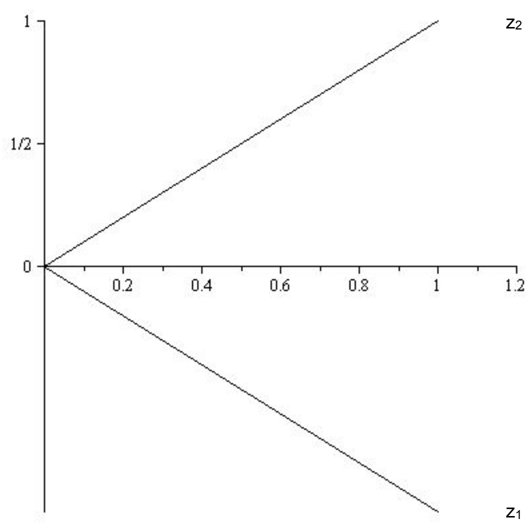


(b)

(c) We have $|z_1| = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$. Also $\arg(z_2) = \arctan(\sqrt{3}) = \frac{\pi}{3}$. Hence, in polar form $z_2 = \cos(\pi/3) + i\sin(\pi/3)$ and $z_1 = \cos(5\pi/3) + i\sin(5\pi/3)$.

Question 1. Version B.

(a) Using the quadratic formula, we have
$z_{1,2} = 1 \pm i$.

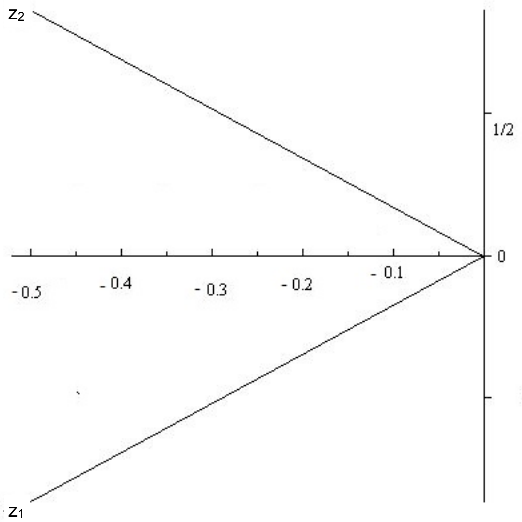


(b)

(c) We have $|z_1| = |z_2| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Also $\arg(z_2) = \arctan(1) = \frac{\pi}{4}$ and since $z_2 = \overline{z_1}$ then $\arg(z_1) = \frac{7\pi}{4}$. Hence, in polar form $z_2 = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$ and $z_1 = \sqrt{2}(\cos(7\pi/3) + i\sin(7\pi/4))$.

Question 1. Version C.

(a) Using the quadratic formula, we have
$z_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.



(b)

(c) We have $|z_1| = |z_2| = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$.
Since $a = -\frac{1}{2} < 0$ and $b = \frac{\sqrt{3}}{2} > 0$ then $\arg(z_1) = \pi - \arctan(\left|\frac{b}{a}\right|) = \pi - \arctan(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and since $z_2 = \overline{z_1}$ then $\arg(z_2) = \frac{4\pi}{3}$. Hence, in polar form $z_2 = \cos(2\pi/3) + i\sin(2\pi/3)$ and $z_1 = \cos(4\pi/3) + i\sin(4\pi/3)$.

Question 2. Version A

Solution A

Question 2. Version B

Solution C

Question 2. Version C

Solution D

Question 3. Version A

$\det(A) = 3 \times 1 - 0 \times 2 = 3 \neq 0$, so A is invertible and A^{-1} is given by
$A^{-1} = \frac{1}{3\times1-0\times2}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix}$

Question 3. Version B

$\det(A) = 2 \times 3 - 1 \times 3 = 3 \neq 0$, so A is invertible and A^{-1} is given by
$A^{-1} = \frac{1}{2\times3-1\times3}\begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1/3 & 2/3 \end{bmatrix}$

Question 3. Version C

$\det(A) = 3 \times 3 - 1 \times 3 = 6 \neq 0$, so A is invertible and A^{-1} is given by
$A^{-1} = \frac{1}{3\times3-3\times2}\begin{bmatrix} 3 & -3 \\ -1 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/6 & 1/2 \end{bmatrix}$

Question 4. Version A

We first find the upper triangular form of the augmented matrix of this system. We have

$$\left(\begin{array}{cc|c} 4 & 2 & 2 \\ 3 & b & c \end{array}\right) \xrightarrow{R_2 \to R_2 - (3/4)R_1} \left(\begin{array}{cc|c} 4 & 2 & 2 \\ 0 & b - 3/2 & c - 3/2 \end{array}\right).$$

(i) To have no solution, the last column has to be a pivot column; i.e. $b - 3/2 = 0$ and $c - 3/2 \neq 0 \Leftrightarrow b = 3/2$ and $c \neq 3/2$.

(ii) For infinitely many solutions, we need fewer pivots than variables; i.e., the second row has to be $[0\ 0\,|\,0] \Leftrightarrow b - 3/2 = 0$ and $c - 3/2 = 0$. Hence $b = c = 3/2$.

(iii) For a unique solution, we the first and second columns have to be pivot columns; i.e., $b - 3/2 \neq 0 \Leftrightarrow b \neq 3/2$ and c arbitrary.

Question 4. Version B

We first find the upper triangular form of the augmented matrix of this system. We have

$$\left(\begin{array}{cc|c} 2 & 3 & c \\ 3 & b & 2 \end{array}\right) \xrightarrow{R_2 \to R_2 - (3/2)R_1} \left(\begin{array}{cc|c} 2 & 3 & c \\ 0 & b - 9/2 & 2 - 3c/2 \end{array}\right).$$

(i) To have no solution, the last column has to be a pivot column; i.e. $b - 9/2 = 0$ et $2 - 3c/2 \neq 0 \Leftrightarrow b = 9/2$ and $c \neq 4/3$.

(ii) For infinitely many solutions, we need fewer pivots than variables; i.e., the second row has to be $[0\ 0\,|\,0] \Leftrightarrow b - 9/2 = 0$ and $2 - 3c/2 = 0, \Leftrightarrow b = 9/2,\ c = 4/3$.

(iii) For a unique solution, we the first and second columns have to be pivot columns; i.e., $b - 9/2 \neq 0 \Leftrightarrow b \neq 9/2$ and c arbitrary.

Question 4. Version C We first find the upper triangular form of the augmented matrix of this system. We have

$$\left(\begin{array}{cc|c} 3 & b & 2 \\ 2 & 3 & c \end{array}\right) \xrightarrow{R_2 \to R_2 - (2/3)R_1} \left(\begin{array}{cc|c} 3 & 2 & 2 \\ 0 & 3 - 2b/3 & c - 4/3 \end{array}\right).$$

(i) To have no solution, the last column has to be a pivot column; i.e., $3 - 2b/3 = 0$ and $c - 4/3 \neq 0 \Leftrightarrow b = 9/2$ and $c \neq 4/3$.

(ii) For infinitely many solutions, we need fewer pivots than variables; i.e., the second row has to be $[0\ 0\,|\,0] \Leftrightarrow 3 - 2b/3 = 0$ and $c - 4/3 = 0$. Hence $b = 9/2,\ c = 4/3$.

(iii) For a unique solution, we the first and second columns have to be pivot columns; i.e. $3 - 2b/3 \neq 0 \Leftrightarrow b \neq 9/2$ and c arbitrary.

Question 5. Version A

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{bmatrix} = (-2-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$
$$= (-1-\lambda)\left[(1-\lambda)^2 - (1)(1)\right] = (\lambda-2)\lambda(-2-\lambda)$$

Hence the eigenvalues are $\lambda_1 = -2,\ \lambda_2 = 0$ and $\lambda_3 = 2$.
Solution: A, D (no part marks)

Question 5. Version B

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ -1 & -1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}$$
$$= (1-\lambda)\left((1-\lambda)(-1-\lambda) - (-1)(1)\right) = \lambda^2(1-\lambda)$$

Hence the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$.
Solution: C, D, E (no part marks)

Question 5. Version C

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 1 & -1-\lambda \end{bmatrix} = (-1-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$
$$= (-1-\lambda)\left((1-\lambda)^2 - (1)(1)\right) = (-1-\lambda)\lambda(\lambda-2)$$

Hence the eigenvalues are $\lambda_1 = -1,\ \lambda_2 = 0$ and $\lambda_3 = 2$.
Solution: A, C (no part marks)

Question 6. Version A

(a) The matrix is $A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$.

$$\det(A - \lambda I_2) = \lambda^2 - 3\lambda = \lambda(\lambda - 3)\,.$$

Hence the eigenvalues are 0 and 3.

To find the eigenvectors, we solve $(A - \lambda I)\vec{x} = \vec{0}$.

- For $\lambda = 0$, we have

$$A - (0)I = A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + (1/2)R_1} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to (1/2)R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

  The eigenvectors are thus $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} r, r \in \mathbb{R}, r \neq 0$.

- For $\lambda = 3$, we have

$$A - (3)I = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

  The eigenvectors are thus $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} s, s \in \mathbb{R}, s \neq 0$.

(b) The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{0t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \text{ where } c_1,\, c_2 \text{ are constants}\,.$$

(c) The particular solution satisfies

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} c_1 - 2c_2 &= 2 \\ c_1 + c_2 &= 5 \end{cases}.$$

Thus $c_1 = 4$ and $c_2 = 1$, so the particular solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - 2e^{3t} \\ 4 + e^{3t} \end{bmatrix}.$$

Question 6. Version B

(a) The matrix is $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

$$\det(A - \lambda I_2) = \lambda^2 + 3\lambda = \lambda(\lambda + 3)\,.$$

Hence the eigenvalues are 0 and -3.

To find the eigenvectors, we solve $(A - \lambda I)\vec{x} = \vec{0}$.

- For $\lambda = 0$, we have

$$A - (0)I_2 = A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to (-1)R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\,.$$

The eigenvectors are thus $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} r, r \in \mathbb{R}, r \neq 0$.

- For $\lambda = -3$, we have

$$A - (-3)I_2 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to (1/2)R_1} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}\,.$$

The eigenvectors are thus $\vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} s, s \in \mathbb{R}, s \neq 0$.

(b) The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{0t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}, \text{ where } c_1,\ c_2 \text{ are constants}\,.$$

(c) The particular solution satisfies

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{rcl} c_1 - (1/2)c_2 & = & 2 \\ c_1 + c_2 & = & 5 \end{array} \right.\,.$$

Thus $c_1 = 3$ and $c_2 = 2$, so the particular solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-3t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - e^{-3t} \\ 3 + 2e^{-3t} \end{bmatrix}\,.$$

Question 6. Version C

(a) The matrix is $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$.

$$\det(A - \lambda I_2) = \lambda^2 + \lambda = \lambda(\lambda + 1)\,.$$

Hence the eigenvalues are 0 and -1.

To find the eigenvectors, we solve $(A - \lambda I)\vec{x} = \vec{0}$.

- For $\lambda = 0$, we have

$$A - (0)I_2 = A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - (1/2)R_1} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to (-1/2)R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\,.$$

  The eigenvectors are thus $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} r, r \in \mathbb{R}, r \neq 0$.

- For $\lambda = -1$, we have

$$A - (-1)I_2 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\,.$$

  The eigenvectors are thus $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} s, s \in \mathbb{R}, s \neq 0$.

(b) The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{0t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \text{ where } c_1,\ c_2 \text{ are constants}\,.$$

(c) The particular solution satisfies

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 &= 2 \\ c_1 + c_2 &= 5 \end{cases}\,.$$

Thus $c_1 = 8$ and $c_2 = -3$, so the particular solution is

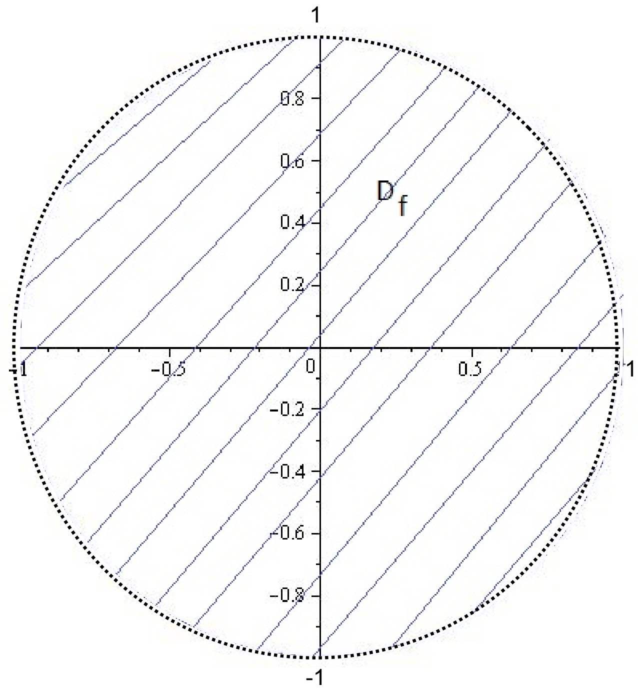
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 - 6e^{-t} \\ 8 - 3e^{-t} \end{bmatrix}\,.$$

Question 7. (a) The domain is

$$D_f = \{(x,y) \in \mathbb{R}^2 | \; 1 - x^2 - y^2 > 0\} = \{(x,y) \in \mathbb{R}^2 | \; x^2 + y^2 < 1\}\,.$$

The range is: $-\infty < f \leq 0$. The range can't be larger than 0 because $x^2 + y^2$ cannot be less than zero.

(b) The domain is the interior of a circle of radius 1. Note that the boundary is not included.



(c)

$$f(x,y) = k \;\Leftrightarrow\; \ln(1 - x^2 - y^2) = k \;\Leftrightarrow\; x^2 + y^2 = 1 - e^k\,.$$

These are circles with radius $1 - e^k$.
For $k_1 = 0$, we have a point $x^2 + y^2 = 0 \;\Rightarrow (x = 0, y = 0)$;
For $k_2 = \ln(3/4)$, we have the circle $x^2 + y^2 = 1/4$;
For $k_3 = \ln(8/9)$, we have the circle $x^2 + y^2 = 1/9$;