University of Ottawa

MAT 1332; Winter 2017, Midterm Exam 1

Wednesday, February 15^{th}, 2017

Instructor (circle one): Guy Beaulieu   Robert Smith?   Xiaoying Wang

DGD (circle one): 1   2   3   4

First Name ____________________

Family Name ____________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more than others. Make note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed.
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X, and Casio FX-300X) are allowed.
- This exam consists of 7 questions: 3 are multiple choice and 4 are long answer.
  - For the 3 multiple choice questions, only the chosen answer will be marked.
  - For the 4 long answer questions, the correct answer requires justification written legibly and logically: you must convince me that you know your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

  By signing below, you acknowledge that you have ensured that you are complying with the above statement.

  Signature :____________________

- Good Luck!

Student Number: ____________________, Total marks: ______________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

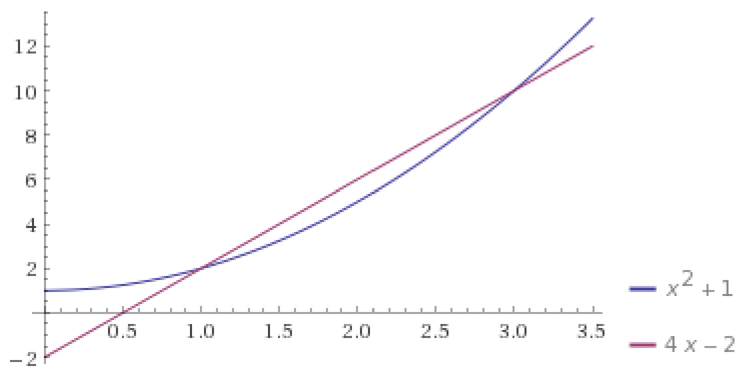
QUESTION 1. (3 points) Determine the area of the region bounded by the graphs of the functions $y = x^2 + 1$ and $y = 4x - 2$.

Among the following answers, circle the correct one.

A) $A = \frac{1}{3}$ (units^2) B) $A = \frac{4}{3}$ (units^2) C) $A = \frac{5}{3}$ (units^2)

D) $A = \frac{2}{3}$ (units^2) E) $A = \frac{7}{3}$ (units^2)

The answer is B.



We find the intersection points between the curves by solving

$$x^2 + 1 = 4x - 2 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3 \text{ or } x = 1$$

Thus, the bounded region is above the interval $[1, 3]$. On this interval $4x - 2 \geq x^2 + 1$, thus the area is computed by the following integral:

$$\begin{aligned} \int_1^3 ((4x-2) - (x^2+1))\, dx &= \int_1^3 (-x^2 + 4x - 3)\, dx \\ &= \left[-\frac{x^3}{3} + 2x^2 - 3x \right]_1^3 \\ &= \frac{4}{3} \end{aligned}$$

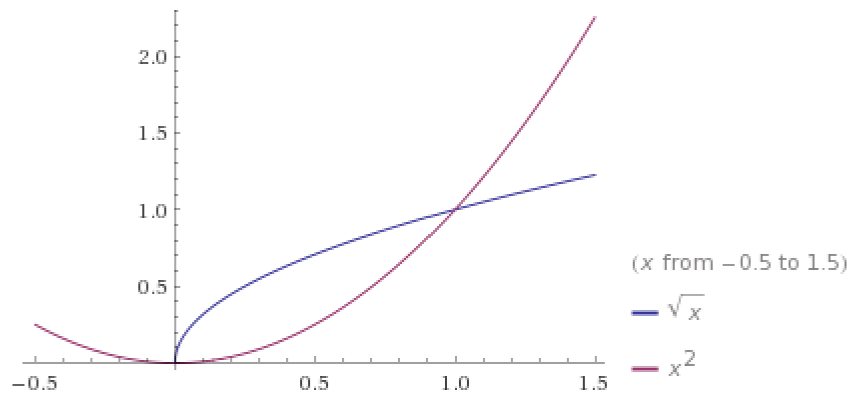
QUESTION 2. (3 points) Determine the volume of the solid generated by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ around the x-axis.

Among the following answers, circle the correct one.

A) $V = \dfrac{3\pi}{5}$ (units^3) B) $V = \dfrac{3\pi}{10}$ (units^3) C) $V = \dfrac{3\pi}{2}$ (units^3)

D) $V = \dfrac{3\pi}{4}$ (units^3) E) $V = \pi$ (units^3)

The answer is B.



We find the intersection points between the curves by solving

$$x^2 = \sqrt{x} \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1$$

Thus, the bounded region that will be rotated is above the interval $[0, 1]$. On this interval the outside radius R is $R = \sqrt{x}$ and the inside radius r is $r = x^2$, thus the volume of the solid generated by the region by a rotation around the x-axis is computed by the following integral:

$$\begin{aligned}
\int_0^1 (\pi R^2 - \pi r^2)\, dx &= \pi \int_0^1 ((\sqrt{x})^2 - (x^2)^2)\, dx \\
&= \pi \int_0^1 (x - x^4)\, dx \\
&= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \\
&= \frac{3\pi}{10}
\end{aligned}$$

QUESTION 3. (3 points) A population of wolves living in a territory grows at a rate given by

$$\frac{dN}{dt} = 2\frac{e^{-\sqrt{t}/2}}{\sqrt{t}},$$

where N is measured in thousands of wolves and t in years.
Determine the total change in the wolf population between $t = 4$ and $t = 16$.

Among the following answers, circle the correct one.

A) $\frac{2}{3}(e^{-2} - e^{-20})$ thousand B) $8(e^{-1} - e^{-2})$ thousand C) $8(e^{-4} - e^{-16})$ thousand

D) $6(e^{-1} - e^{-2})$ thousand E) $\frac{5}{3}(1 - e^{-15})$ thousand

The answer is B.
The change in the population is calculated by the following definite integral:

$$N(16) - N(4) = \int_4^{16} \frac{dN}{dt}\, dt$$

Where by using the substitution with $u = -\frac{\sqrt{t}}{2}$ which implies $dt = -4\sqrt{t}du$ we have that

$$\begin{aligned} \int 2\frac{e^{-\sqrt{t}/2}}{\sqrt{t}}\, dt &= -8\int e^u\, du \\ &= -8e^u + C \\ &= -8e^{-\frac{\sqrt{t}}{2}} \end{aligned}$$

Thus,

$$N(16) - N(4) = (-8e^{-2}) - (-8e^{-1}) = 8(e^{-1} - e^{-2})$$

QUESTION 4. (6 points) Consider the rational function

$$f(x) = \frac{x^2 - 4x}{x^2 - 3x + 2}$$

(a) Decompose f into partial fractions.

(b) Compute the integral $\int f(x)dx$.

(a) After polynomial division we get

$$\frac{x^2 - 4x}{x^2 - 3x + 2} = 1 - \frac{x+2}{(x-2)(x-1)}$$

(2 marks for long division)

Moreover, we have that

$$\frac{x+2}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow x + 2 = A(x-1) + B(x-2)$$

which gives $A = 4$ and $B = -2$. Thus

$$f(x) = 1 - \frac{4}{x-2} + \frac{2}{x-1}$$

(2 marks for partial fractions; 0 marks if partial fractions were applied without dividing first; it is impossible for the quadratic numerator to equal a linear decomposition)

(b)

$$\int f(x)\, dx = x - 4\ln|x-2| + 2\ln|x-1| + C$$

(2 marks: 0.5 for each component, subtract 0.5 for each missing absolute value sign and subtract 0.5 if missing the $+C$)

QUESTION 5. (5 points) The growth of a population of a certain species of fish is modelled by a logistic differential equation to which we add the effect of the presence of a predator, giving the following

$$\frac{dN}{dt} = 0.15N(400 - N) - 0.3N$$

(a) Fnd all biologically reasonable equilibrium points and determine their stability.

(b) Sketch the phase-line diagram for this model.

(a) The equilibrium points are the values of N such that

$$g(N) = 0.15N(400 - N) - 0.3N = 0$$

which gives $N_1 = 0$ (trivial) or $N_2 = 398$.

$$g'(N) = 59.7 - 0.3N$$

When $N_1 = 0$, $g'(0) = 59.7 > 0$ thus this equilibrium point is unstable. When $N_2 = 398$, $g'(398) = -59.7 < 0$ thus this equilibrium point is stable.

(3 marks: 0.5 for each equilibrium, 1 for the derivative (or graph), 0.5 each for determining stability of each equilibrium)

(b)

$$\overset{0}{\bullet} - - - \rightarrow - - - \overset{398}{\bullet} - - - \leftarrow - - - \quad N$$

(2 marks for the graph)

(Life Sciences penalty: subtract 2 marks if arrows are drawn below 0, even if this results in negative marks. You cannot have a negative number of fish.)

QUESTION 6. (5 points) Consider the following intergal:

$$\int_0^2 \frac{2x dx}{(x^2-1)^{1/3}}$$

(a) Explain why this integral is improper.

(b) Determine if it is convergent or divergent. If it converges, compute to which value.

(a) This integral is improper since $f(x) = \frac{2x}{(x^2-1)^{1/3}}$ is undefined at $x = 1$.

(1 mark)

(b)

$$\begin{aligned}
\int_0^2 \frac{2x}{(x^2-1)^{1/3}}\,dx &= \lim_{a\to 1^-}\int_0^a \frac{2x}{(x^2-1)^{1/3}}\,dx + \lim_{b\to 1^+}\int_b^2 \frac{2x}{(x^2-1)^{1/3}}\,dx \\
&= \lim_{a\to 1^-}\left[\frac{3}{2}(x^2-1)^{2/3}\right]_0^a + \lim_{b\to 1^+}\left[\frac{3}{2}(x^2-1)^{2/3}\right]_b^2 \\
&= \lim_{a\to 1^-}\left(\frac{3}{2}(a^2-1)^{2/3} - \frac{3}{2}\right) + \lim_{b\to 1^+}\left(\frac{3}{2}(3)^{2/3} - \frac{3}{2}(b^2-1)^2\right) \\
&= -\frac{3}{2} + \frac{3}{2}(3)^{2/3} \\
&\approx 6.29
\end{aligned}$$

Thus this integral converges to the value $-\frac{3}{2} + \frac{3}{2}(3)^{2/3}$.

(0.5 marks for splitting the integral at $x = 1$, 1 mark for taking left and right limits, 1 for the answer, 0.5 marks for stating the word "converges" (or "convergent") and 1 mark for the answer)

Question 7. (5 points) Determine the average of the function

$$f(t) = 68 + \sin(\frac{\pi t}{12})$$

on the interval $[0, 12]$.

$$\begin{aligned}
avg_f &= \frac{1}{12-0}\int_0^{12} \left(68 + \sin\left(\frac{\pi t}{12}\right)\right)\, dt \\
&= \frac{1}{12}\left[68t - \frac{12}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^{12} \\
&= \frac{1}{12}\left(\left((68)(12) + \frac{12}{\pi}\right) - \left(-\frac{12}{\pi}\right)\right) \\
&= 68 + \frac{1}{\pi} + \frac{1}{\pi} \\
&= 68 + \frac{2}{\pi} \\
&\approx 68.3183 \text{ units}
\end{aligned}$$

(2 marks for the denominator, 2 marks for the answer, 1 mark for "units")